Artificial Intelligence

Lecture 16

Symbolic Logic

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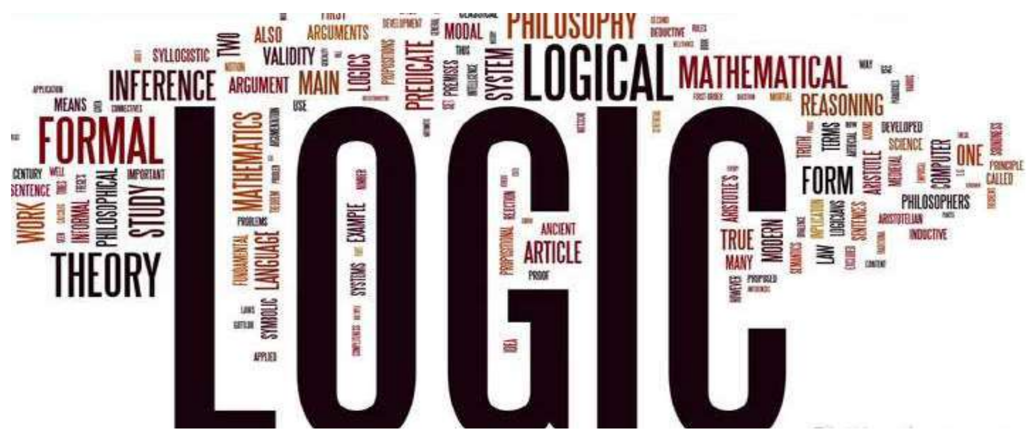
Symbolic Logic

Lecture Outlines:

- …
- First Order Predicate Logic (FOPL)
- Syntax and Semantics of FOPL
- WFFs and Properties of wffs
- Clausal Form and Resolution
- Conversion to Clausal Form



First Order Predicate Logic (FOPL)

- **FOPL** extends the expressiveness of PL (**Propositional Logic**).
- It is a generalization of PL that permits reasoning about world objects as relational entities as well as classes or subclasses of objects.
- The generalization comes from:
  - *the introduction of predicates in place of propositions,*
  - *the use of functions and*
  - *the use of variables together with variable quantifiers.*

Syntax of FOPL

- The syntax of FOPL is determined by allowable symbols and rules of combination, are defined as follows:
  - **Connectives**: Five connective symbols, ~ (not or negation), & (and or conjunction), V (or or disjunction), → (implication), ↔ (equivalence or if and only if).
  - **Quantifiers**
  - **Constants**
  - **Variables**
  - **Functions**
  - **Predicates**

- Two quantifiers: universal and existential quantifiers
- Constants, variables, and functions are referred to as terms.
- Predicates are referred to as atomic formulas or atoms.

Expressions in FOPL

- Consider the following statements:

  E1: All employees earning $1400 or more per year pay tax.

  E2: Some employees are sick today.

  E3: No employee earns more than the president.

- To represent expressions in FOPL, we must define predicates and functions, such as-

  E(x) for x is an employee.

  P(x) for x is president.

  i(x) for the income of x.

  GE(u,v) for u is greater or equal to v.

  S(x) for x is sick today.

  T(x) for x pays taxes.

Expressions in FOPL

- Now we can represent E1, E2 and E3 as:
- $E1' = \forall x((E(x) \& GE(i(x), 1400)) \rightarrow T(x))$
- $E2' = \exists y(E(y) \rightarrow S(y))$
- $E3' = \forall xy((E(x) \& P(y)) \rightarrow GE(i(x), i(y)))$
- The expressions $E1'$, $E2'$ and $E3'$ are known as well-formed formulas or wffs. Wffs are defined as follows:
  - An atomic formula is a wff.
  - If P and Q are wffs, then ~P, P&Q, PVQ, P→Q, PQ, $\forall x$, $P(x)$, and $\exists x P(x)$ are wffs.

Properties of wffs

- **Valid:** A wff is said to be valid if it is true under every interpretation.
- **Inconsistent or unsatisfiable:** A wff that is false under every interpretation is said to inconsistent or unsatisfiable.
- **Invalid:** A wff that is not valid (false for some interpretation) is invalid.
- **Satisfiable:** A wff that is not inconsistent (true for some interpretation) is satisfiable.
- *A valid wff is satisfiable and an inconsistent wff is invalid.*

Semantics of FOPL

- The semantics of FOPL are determined by interpretations assign to predicates, rather than propositions.
- This means that an interpretations must also assign values to other terms including constants, variables and functions.
  - When an assignment of values is given to each term and to each predicate symbol in a wff, we say an interpretation is given to the wff.
  - The value of any given wff can be determined by truth table.
  - If the truth values for two wffs are the same under every interpretation, they are equivalent.
- A predicate (or wff) has no variables is called a ground atom.

Semantics of FOPL

- **Example:**

  Evaluate E: $\forall x((A(a,x) V B(f(x)) \& C(x)) \rightarrow D(x)$

- Here, four predicates, A, B, C and D. A is a two-place predicate, a is constant and x is variable.
- B, C and D are unary predicates, the arguments of B is a function f(x).
- E is quantified by the universal quantifier (for all x).

Semantics of FOPL

- **Example:**

  Evaluate E: $\forall x((A(a,x) V B(f(x)) \& C(x)) \rightarrow D(x)$

- To evaluate E, consider the domain value {1, 2} and the following values:

a = 2 f(1)=2 f(2)=1

A(2,1) = true A(2,2) = false B(1) = true B(2) = false

C(1) = true C(2) = false D(1) = false D(2) = true

Using the above values we can evaluate E:

1) For x=1, we get true ---> false; So, E equals to False.

2) For x=2, E equals to True.

Since E is not true for all x, the expression E evaluates to False.

Equivalent Logical Expressions

- $PV(Q\&R) = (PVQ) \& (PVR)$
- $\sim(P\&Q) = \sim PV\sim Q$
- $P\rightarrow Q = \sim P \, V \, Q$
- $P\leftrightarrow Q = (P \rightarrow Q)\&(Q \rightarrow P)$
- $\forall x P(x) V Q = \forall x(P(x) V Q)$
- $\exists x P(x) \& Q = \exists x(P(x) \& Q)$

- **Theorem-1:** A wff Q is a logical consequence of the wffs p_1, p_2, …, p_n if and only if whenever $p_1 \& p_2 \& \ldots \& p_n$ is true under an interpretation, Q is also true.

Clausal Form (Normal Form)

- A clause can be defined as a wff consisting of disjunction of literals, i.e., $P_1 \vee P_2 \vee P_3 \ldots \vee \sim P_n$, where P_1, P_2, …, P_n are literals.

- **Conjunctive Normal Form (CNF)**
- **Disjunctive Normal Form (DNF)**
- **Prenex Normal Form**

Clausal Form (Normal Form)

- **Conjunctive Normal Form (CNF)**

If W1, W2, ..., Wn each possibly consisting of disjunction of literals, then we say W1 $\wedge$ W2 $\wedge$ ... $\wedge$ Wn is in *conjunctive normal form*, e.g. $(\neg P \vee Q) \wedge (P \vee \neg Q \vee R) \wedge \neg R$ is in CNF

- **Disjunctive Normal Form (DNF)**

If W1, W2, ..., Wn each possibly consisting of conjunction of literals, then we say W1 $\wedge$ W2 $\vee$ ... $\vee$ Wn is in *disjunctive normal form*, e.g. $(\neg P \wedge Q) \vee (P \wedge \neg Q \wedge R) \vee \neg R$ is in DNF.

Note: Any wff can be transformed into either normal form.

e.g. consider $(P \wedge Q) \vee (R \wedge S)$ is in DNF and can be transformed into CNF using the equivalent logical expressions as follows:

$((P \wedge Q) \vee R) \wedge (P \wedge Q) \vee S)$
$((P \vee R) \wedge (Q \vee R)) \wedge ((P \vee S) \wedge (Q \vee S))$
$(P \vee R) \wedge (Q \vee R) \wedge (P \vee S) \wedge (Q \vee S)$

Clausal Form (Normal Form)

Prenex Normal Form:

A formula of first-order logic is in prenex normal form if it is of the form

$$Q_1 x_1 \ldots Q_n x_n M,$$

where each Q_i is a quantifier $\forall$ ("for all") or $\exists$ ("exists") and M is quantifier-free.

For example, the formula

$$\exists x \forall y \exists z (P(x) \vee Q(x, y, z))$$

is in prenex normal form, whereas formula

$$\exists x \forall y (P(x) \vee \exists z Q(x, y, z))$$

is not, where $\vee$ denotes OR.

Every formula of first-order logic can be converted to an equivalent formula in prenex normal form.

Resolution: Conversion to Clausal Form

Resolution is an important rule of inference, that can be applied to a set of clauses. This method was developed by J. A. Robinson in 1965. For applying resolution, we have to convert all the wffs to clausal form. The conversion procedure is given below:

1. ***Eliminate all implication symbols.***
   $A \rightarrow B$ with $\neg A \vee B$ and
   $A \leftrightarrow B$ with $(\neg A \vee B) \wedge (\neg B \vee A)$
2. ***Reduce the scope of negation.*** (Using D'Morgan's law and other equivalences). Aim is to apply each negation symbol to atmost one atomic formula.

- ~(P & Q) = ~P V ~Q
- ~(P V Q) = ~P & ~Q
- P → Q = ~P V Q
- P⇔Q = (P → Q) & (Q → P) = (~P V Q) & (~Q V P)

Resolution: Conversion to Clausal Form

3. ***Standardize variables.*** Within the scope of a quantifier a variable bound by quantifier is a dummy variable. It can be replaced by any other variable, which has not occured anywhere else throughout the scope of the quantifier without changing the truth value of the wff.

   This is to ensure that each quantifier has its own unique dummy variable. e.g. $\forall x\ P(x) \rightarrow \exists x\ Q(x)$ can be writtten as $\forall x\ P(x) \rightarrow \exists y\ Q(y)$.

4. ***Eliminate existential quantifier.*** This is to replace each occurrence of existentially quantified variables by a Skolem function whose arguments are those universally quantified variables, whose scope includes the scope of the existential quantifier.

   For example, consider the wff, $\forall x\ (\exists y\ P(x, y))$; here existence of y depends on the value of x, which is explicitly defined by some function g(x), which maps each x into y. Thus we can replace y with g(x) in the previous wff, which will become $\forall x\ P(x, g(y))$. (This process is called Skolemization).

Resolution: Conversion to Clausal Form

5. ***Convert to prenix form.*** In this stage there are no E and each A has its own variable. So we can move A to the beginning.
6. ***Convert to CNF.***
7. ***Eliminate.*** In this stage all the variables we are using are universally quantified. So we can eliminate the explicit occurrence of A. Now we are left with a matrix in CNF form.
8. ***Eliminate ∧ symbols.*** It is done by writing down the expression of the form (Wl ∧ W2 ∧ ... ∧ Wn) as {Wl, W2, ..., Wn}. Here each Wi, i = 1, 2, ..., n will be a disjunction of literals.
9. ***Rename variables.*** Variable symbol may be renamed so that no variable symbol appears more than one clause.

Resolution: Conversion to Clausal Form

e.g.: $S = \forall x \{P(x) \rightarrow \{\forall y[P(y) \rightarrow P(f(x,y))] \wedge \neg(\forall y)[Q(x,y) \rightarrow P(y))]\}\}$

1. $\forall x \{\neg P(x) \vee \{\forall y[\neg P(y) \vee P(f(x,y))] \wedge \neg(\forall y)[\neg Q(x,y) — P(y)]\}\}$
2. $\forall x \{\neg P(x) \vee \{\forall y[\neg P(y) \vee P(f(x,y))] \wedge \exists y[Q(x,y) \wedge \neg P(y)]\}\}$ D'Morgan's Law
   (By using $\neg(\forall x)P(x) = \exists x (\neg P(x)$ and D'Morgan's Law)
3. $\forall x \{\neg P(x) \vee \{\forall y[\neg P(y) \vee P(f(x,y))] \wedge \exists w[Q(x,w) \wedge \neg P(w))]\}\}$ Rename variables
4. $\forall x \{\neg P(x) \vee \{\forall y[\neg P(y) \vee P(f(x,y))] \wedge [Q(x, g(x)) \wedge \neg P(g(x))]\}\}$ Remove Existential Q
5. $\forall x \forall y \{\neg P(x) \vee \{[\neg P(y) \vee P(f(x,y))] \wedge [Q(x,g(x)) \wedge \neg P(g(x))]\}\}$ Prenex form
6. $\forall x \forall y \{[\neg P(x) \vee \neg P(y) \vee P(f(x,y))] \wedge [\neg P(x) \vee Q(x,g))] \wedge [\neg P(x) \vee \neg P(g(x))]\}$
7. $\{[\neg P(x) \vee P(y) \vee P(f(x,y))] \wedge [\neg P(x) \vee Q(x,g(x))] \wedge [\neg P(x) \vee \neg P(g(x))]\}$
8. $\{\neg P(x) \vee \neg P(y) \vee P(f(x,y)),$
   $\neg P(x) \vee Q(x,g)),$
   $\neg P(x) \vee \neg P(g(x))\}$
9. $\{\neg p(x1) \vee \neg P(y) \vee P(f(x1,y)),$
   $\neg P(x2) \vee Q(x2,g(x2)),$
   $\neg P(x3) \vee \neg P(g(x3))\}$

Symbolic Logic

THE END