Complex variable
Laplace transform
and
Fourier analysis

Complex Number

Definition of Complex Number: Let x and y be two real numbers. Then a number of the form $x + iy = z$ is called a complex number where $i = \sqrt{-1}$.

And x is called the real part of the complex number z and y is called the imaginary part of z and these are denoted by $Re\, z$ and $Im\, z$ respectively.

Note: The complex number $x + iy$ *is* also represented by the ordered pair of real numbers (x, y), *i.e.,* $x + iy = (x, y)$.

\# Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 \leq z_2$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 x_2 + y_1 y_2)$$

Definition: Let the complex number $z = x + iy$. Then the conjugate of z is $x - iy$ and is denoted by $\bar{z}$ i.e., if $z = x + iy$ then $\bar{z} = x - iy$.

Theorem: The sum and product of a complex number and its conjugate are real numbers.

Proof: Let $x + iy$ be a complex number, then its conjugate is $x - iy$.

Now Sum= $(x + iy)$ +$(x - iy) = 2x$ (real)

Product=$(x + iy)(x - iy) = x^2 + y^2$ (real)

\# A complex number $z = x + iy$ represents a point in the complex plane i.e., R^2 plane (space) as shown in the figure 1.

\# If $z = x + iy$ is a complex number denoted by the point P then r, the length of OP where O is the origin is called the modulas of z and is denoted by $|z|$.

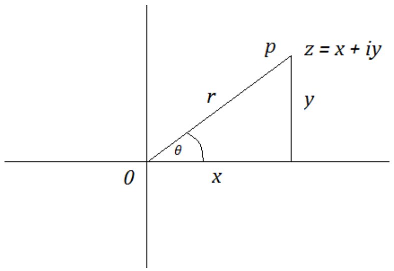


Figure 1

i.e., $r = |z| = \sqrt{x^2 + y^2}$.

And if θ is the angle between the line OP and the real axis then θ is called argument or amplitude of z and is denoted by $\arg z$ or amp z.

i.e., $$\theta = \arg z = tan^{-1}\left(\frac{y}{x}\right)$$

From the figure

$$x = r\, cos\theta$$

$$y = r \sin\theta$$

Hence $$z = r\, cos\,\theta + ir\, sin\,\theta$$

$$= r\,(cos\,\theta + isin\,\theta)$$

$$= re^{i\theta} \text{ [polar form of } z]$$

Similarly $$\bar{z} = \mathrm{re}^{-i\theta}$$

Therefore arg $\bar{z} = -\arg z$

And $|z| = |\bar{z}|$

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$|z_1 z_2| = |r_1 r_2| = |z_1||z_2|$$

And, $$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_{1)}) + \arg(z_2)$$

Similarly,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{provided } |z_2| \neq 0$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_{1)}) - \arg(z_2)$$

Problem: If $x = cos\theta + i\, sin\theta$ and $x = cos\phi + i\, sin\phi$, prove that

$$\frac{x+y}{x-y} = i \tan\left(\frac{\theta - \phi}{2}\right)$$

De Moivere's theorem (By Exponential Function)**:** $(cos\theta + i\, sin\theta)^n = \cos n\theta + i \sin n\theta$

Proof: We know that $cos\theta + i\, sin = e^{i\theta}$

Therefore $(cos\theta + i \sin\theta)^n$ $= (e^{i\theta})^n$
$= e^{in\theta}$
$= \cos n\theta + i \sin n\theta$

Example: Express $\frac{(cos\theta + i\, sin\theta)^8}{(sin\theta + i\, cos\theta)^4}$ in the form $x + iy$.

Soln: $$\frac{(cos\theta + i\, sin\theta)^8}{(sin\theta + i\, cos\theta)^4} = \frac{(cos\theta + i\, sin\theta)^8}{i^4\left(cos\theta + \frac{1}{i}sin\theta\right)^4}$$
$$= \frac{(cos\theta + i\, sin\theta)^8}{(cos\theta - i\, sin\theta)^4}$$
$$= \frac{(cos\theta + i\, sin\theta)^8}{(\cos(-\theta) + i\sin(-\theta))^4}$$
$$= \frac{(cos\theta + i\, sin\theta)^8}{(cos\theta + i\, sin\theta)^{-4}}$$
$$= (cos\theta + i\, sin\theta)^{12}$$
$$= \cos 12\theta + i \sin 12\theta \quad \text{(Proved)}$$

\# $|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}$

\# $\overline{z_1 \pm z_2} = \overline{(x_1 \pm x_2) + \iota(y_1 \pm y_2)} = (x_1 \pm x_2) - i(y_1 \pm y_2)$

$$= (x_1 - iy_1) \pm (x_2 - iy_2) = \overline{z_1} \pm \overline{z_2}$$

And, $\overline{z_1 z_2} = \overline{(x_1 + \iota y_1)(x_2 + \iota y_2)} = \overline{(x_1x_2 - y_1y_2) + \iota(x_1y_2 + x_2y_1)}$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$

$= (x_1 - iy_1)(x_2 - iy_2) = \bar{z}_1\bar{z}_2$

And $\quad \bar{\bar{z}} = z$

Definition of Complex Variable: A variable z which takes any value from a set of complex numbers is called complex variable.

Problem: For any two complex number z_1 and z_2, prove that

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$
(ii) $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof:

(i) We have,

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} \qquad \because z_1\bar{z}_1 = |z_1|^2$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} \qquad \because z + \bar{z} = 2x = 2Re(z)$

$= |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z}_2)$z=x+iy

$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$ Re z =x≤√(x²+y²)=|z|

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$= \{|z_1| + |z_2|\}^2$

So, $|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

Hence, $|z_1 - z_2| \geq |z_1| - |z_2|$

Example: Show that $|x| + |y| \leq \sqrt{2}|x + iy|$

Solution: We know,

$$2|x + iy|^2 = 2(x^2 + y^2) = x^2 + y^2 + x^2 + y^2 = |x|^2 + |y|^2 + |x|^2 + |y|^2$$

$$\geq |x|^2 + |y|^2 + 2|x||y| \qquad \because x^2 + y^2 \geq 2xy \text{ i.e., AM} \geq \text{GM}$$

$$\geq (|x| + |y|)^2$$

Hence, $|x| + |y| \leq \sqrt{2}|x + iy|$

Powers of i: We know that $i = \sqrt{-1}$

Therefore $\quad i^2 = -1$

$$i^3 = i^2 i = -1i = -i$$

$$i^4 = i^3 i = -ii = -i^2 = -(-1) = 1$$

Limit

Definition: Let A and B be two sets. A function $f: A \to B$ is a correspondence such that by f each elements of A corresponds to a unique element of B.

Here we shall always consider the function $f: C \to C$, where C is the set of complex numbers.

Definition: Let $f(z)$ be a function. A number l is called the limit of $f(z)$ at $z = z_0$ if for any $\varepsilon > 0$

There exists a $\delta > 0$ (depending on ε) such that $|f(z) - l| < \varepsilon$ where $0 < |z - z_0| < \delta$.

i.e., $f(z) \to l$ as $z \to z_0$

$$\lim_{z \to z_0} f(z) = l$$

Example: prove that $\lim_{z \to z_0} z^2 = z_0^2$

Proof: Here $f(z) = z^2$

Choose any $\varepsilon > 0$

$\lim_{z \to z_0} f(z) = z_0^2$ is true if there exists a $\delta > 0$ such that $|f(z) - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

Consider $\delta < 1 \qquad \therefore |z - z_0| < \delta < 1$

Now, $|f(z) - z_0^2| = |z^2 - z_0^2| = |(z + z_0)(z - z_0)|$

$= |(z - z_0 + 2z_0)(z - z_0)|$

$< \delta|(z - z_0) + 2z_0| \qquad \because |z - z_0| < \delta$

$\leq \delta(|z - z_0| + 2|z_0|)$

$< \delta(1 + 2|z_0|) \qquad \because |z - z_0| < \delta < 1$

Suppose $\delta = \min\left(1, \frac{\varepsilon}{1 + 2|z_0|}\right)$

Then $|f(z) - z_0^2| < \frac{\varepsilon}{1+2|z_0|}(1 + 2|z_0|) = \varepsilon$ whenever $0 < |z - z_0| < \delta$

Hence, $\lim_{z \to z_0} f(z) = z_0^2$

$\lim_{z \to z_0} z^2 = z_0^2$ [Proved]

Similarly we can show that $\lim_{z \to z_0} z^n = z_0^n$

Theorem: If $\lim_{z \to z_0} f(z)$ exists then its value is unique.

Proof: Suppose $\lim_{z \to z_0} f(z) = l_1$ and $\lim_{z \to z_0} f(z) = l_2$

We have to show that $l_1 = l_2$.

Choose $\varepsilon > 0$. Then there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(z) - l_1| < \frac{\varepsilon}{2}$, whenever $0 < |z - z_0| < \delta_1$

and $|f(z) - l_2| < \frac{\varepsilon}{2}$, whenever $0 < |z - z_0| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$

Hence, for $0 < |z - z_0| < \delta$

$|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2|$

$\leq |l_1 - f(z)| + |f(z) - l_2|$

$= |f(z) - l_1| + |f(z) - l_2|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Or, $|l_1 - l_2| < \varepsilon$

Since ε is arbitrary,

$|l_1 - l_2| = 0 \quad or \quad l_1 - l_2 = 0 \quad or \quad l_1 = l_2$

Properties: Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then:

(i) $\lim_{z\to z_0}\{f(z) \pm g(z)\} = A \pm B$

(ii) $\lim_{z\to z_0} f(z).g(z) = A.B$

(iii) $\lim_{z\to z_0} \frac{1}{g(z)} = \frac{1}{B}$ provided $B \neq 0$

(iv) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ provided $B \neq 0$

Proof:

(i) Choose $\varepsilon > 0$

Since $\lim\limits_{z\to z_0} f(z) = A$, so there exist $\delta_1 > 0$such that $|f(z) - A| < \frac{\varepsilon}{2}$ whenever $0 < |z - z_0| < \delta_1$

Again since $\lim\limits_{z\to z_0} g(z) = B$, so there exist $\delta_2 > 0$ such that $|g(z) - B| < \frac{\varepsilon}{2}$ whenever $0 < |z - z_0| < \delta_2$

Let, $\delta = \min\,(\delta_1, \delta_2)$

So, for $0 < |z - z_0| < \delta$

$|f(z) + g(z) - (A + B)| = |\{f(z) - A\} + \{g(z) - B\}|$

$\leq |f(z) - A| + |g(z) - B|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence, $\lim_{z\to z_0}\{f(z) + g(z)\} = A + B$

Similarly, $\lim_{z\to z_0}\{f(z) - g(z)\} = A - B$

(ii) We get,

$|f(z)g(z) - AB| = |f(z)g(z) - f(z)B + f(z)B - AB|$

$= |f(z)\{g(z) - B\} + B\{f(z) - A\}|$

$\leq |f(z)||g(z) - B| + |B||f(z) - A|$

Since, $\lim_{z\to z_0} f(z) = A$, So there exists $\delta_1 > 0$ such that $|f(z) - A| < 1$ whenever $0 < |z - z_0| < \delta_1$

But $|f(z)| - |A| \leq |f(z) - A|$

$\therefore |f(z)| - |A| < 1$

Or $|f(z)| < 1 + |A|$

Since $\lim\limits_{z\to z_0} g(z) = B$, so there exists $\delta_2 > 0$ such that $|g(z) - B| < \frac{\varepsilon}{2(1+|A|)}$ whenever $0 < |z - z_0| < \delta_2$.

Since $\lim\limits_{z\to z_0} f(z) = A$, so there exists $\delta_3 > 0$ such that $|f(z) - A| < \frac{\varepsilon}{2(|B|)}$ whenever $0 < |z - z_0| < \delta$.

Let $\delta = \min\,(\delta_1, \delta_2, \delta_3)$

So for $0 < |z - z_0| < \delta$

$|f(z)g(z) - AB| < (1 + |A|)\dfrac{\varepsilon}{2(1 + |A|)} + |B|\dfrac{\varepsilon}{2|B|}$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$|f(z)g(z) - AB| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

$\therefore \lim\limits_{z\to z_0} f(z)g(z) = AB$

(iii) We get,

$$\left|\frac{1}{g(z)} - \frac{1}{B}\right| = \frac{|g(z) - B|}{|B||g(z)|}$$

Since $\lim\limits_{z\to z_0} g(z) = B \neq 0$, so there exists $\delta_1 > 0$ such that $|g(z) - B| < \frac{|B|^2\varepsilon}{2}$for $0 < |z - z_0| < \delta_1$

Again since $\lim\limits_{z\to z_0} g(z) = B$ so there exists $\delta_2 > 0$ such that $|g(z) - B| < \frac{|B|}{2}$ whenever $0 < |z - z_0| < \delta_2$.

But $B = B - g(z) + g(z)$

$\therefore |B| \leq |g(z) - B| + |g(z)|$

$< \frac{|B|}{2} + |g(z)|$

$or\ |g(z)| > \dfrac{1}{2}|B|$

$or\ \dfrac{1}{|g(z)|} < \dfrac{2}{|B|}$

Let $\delta = \min\,(\delta_1, \delta_2)$

So for $0 < |z - z_0| < \delta$ we get,

$$\left|\frac{1}{g(z)} - \frac{1}{B}\right| = \frac{|g(z) - B|}{|B|}.\frac{1}{|g(z)|}$$

$< \frac{|B|^2\varepsilon}{2}\,\frac{1}{|B|}.\frac{2}{|B|} = \varepsilon$

Or $\left|\frac{1}{g(z)} - \frac{1}{B}\right| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

Therefore $\lim\limits_{z\to z_0} \frac{1}{g(z)} = \frac{1}{B}$

(iv) $\lim\limits_{z\to z_0} \frac{f(z)}{g(z)} = \lim\limits_{z\to z_0} \left\{f(z).\frac{1}{g(z)}\right\} = \lim\limits_{z\to z_0} f(z).\lim\limits_{z\to z_0} \frac{1}{g(z)}$

$= A.\frac{1}{B}$ [By (ii) and (iii)]

$= \frac{A}{B}$ [Proved]

<u>**Example:**</u> **Show that $\lim\limits_{z\to z_0} \frac{\bar{z}}{z}$ does not exist**

<u>Solution:</u> Let $z = x + iy$

Then$\lim\limits_{z\to 0}\frac{\bar{z}}{z} = \lim\limits_{(x,y)\to(0,0)}\frac{x-iy}{x+iy}$

Suppose z→0 along x axis (y=0, x→0)

Then $\lim\limits_{z\to 0}\frac{\bar{z}}{z} = \lim\limits_{x\to 0, y=0}\frac{x-iy}{x+iy} = \lim\limits_{x\to 0}\frac{x}{x} = 1$

Now let z→0 along y axis (x=0, y→0)

Then $\lim\limits_{z\to 0}\frac{\bar{z}}{z} = \lim\limits_{x=0, y\to 0}\frac{x-iy}{x+iy} = \lim\limits_{x\to 0}\frac{-iy}{iy} = -1$

Since the two limits are unequal, so $\lim\limits_{z\to z_0}\frac{\bar{z}}{z}$does not exist.

Continuity

Definition: A function $f(z)$ is said to be continuous at z_0 if for any $\varepsilon > 0$ there exist $\delta > 0$ (depending on ε and also on z_0) such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ or if $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

Definition: A function $f(z)$ is said to be uniformly continuous in a region R if for any $\varepsilon > 0$ there exist $\delta > 0$ (depending on ε, not at any point in R) such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$ in R where $z_1, z_2 \in R$

Properties: If $f(z)$ and $g(z)$ are continuous at $z_0$then,

(i) $f(z) \pm g(z)$ is continuous at z_0
(ii) $f(z)g(z)$ is continuous at z_0
(iii) $\frac{f(z)}{g(z)}$ is continuous at z_0, provided $g(z) \neq 0$

Proof: (i) Choose $\varepsilon > 0$

Since $f(z)$ is continuous at z_0, then $\lim\limits_{z\to z_0} f(z) = f(z_0)$. So there exist $\delta_1 > 0$such that $|f(z) - f(z_0)| < \frac{\varepsilon}{2}$ whenever $|z - z_0| < \delta_1$

Again since $g(z)$ is continuous at z_0, then $\lim\limits_{z\to z_0} g(z) = g(z_0)$. So there exist $\delta_2 > 0$ such that $|g(z) - g(z_0)| < \frac{\varepsilon}{2}$ whenever $|z - z_0| < \delta_2$.

Let, $\delta = \min(\delta_1, \delta_2)$

So, for $|z - z_0| < \delta$

$$|f(z) + g(z) - (f(z_0) + g(z_0))| = |\{f(z) - f(z_0)\} + \{g(z) - g(z_0)\}|$$
$$\leq |f(z) - f(z_0)| + |g(z) - g(z_0)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $\lim_{z\to z_0}\{f(z) + g(z)\} = f(z_0) + g(z_0)$

Therefore $f(z) + g(z)$ is continuous at z_0.

Similarly we can show that $f(z) - g(z)$ is continuous at z_0.

(ii) We get

$$|f(z)g(z) - f(z_0)g(z_0)| = |f(z)g(z) - f(z)g(z_0) + f(z)g(z_0) - f(z_0)g(z_0)|$$
$$= |f(z)\{g(z) - g(z_0)\} + g(z_0)\{f(z) - f(z_0)\}|$$
$$\leq |f(z)||g(z) - g(z_0)| + |g(z_0)||f(z) - f(z_0)|$$

Since $f(z)$ is continuous at z_0, then $\lim\limits_{z\to z_0} f(z) = f(z_0)$. So there exist $\delta_1 > 0$ such that $|f(z) - f(z_0)| < 1$ whenever $|z - z_0| < \delta_1$

But $|f(z)| - |f(z_0)| \leq |f(z) - f(z_0)|$
$\therefore |f(z)| - |f(z_0)| < 1$
Or, $|f(z)| < 1 + |f(z_0)|$

Again since $g(z)$ is continuous at z_0, then $\lim_{z\to z_0} g(z) = g(z_0)$. So there exist $\delta_2 > 0$ such that $|g(z) - g(z_0)| < \frac{\varepsilon}{2(1+|f(z_0)|}$ whenever $|z - z_0| < \delta_2$.

Further Since $f(z)$ is continuous at z_0, then $\lim_{z\to z_0} f(z) = f(z_0)$. So there exist $\delta_3 > 0$ such that $|f(z) - f(z_0)| < \frac{\varepsilon}{2|g(z_0)|}$ whenever $|z - z_0| < \delta_3$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$
So for $|z - z_0| < \delta$

$$|f(z)g(z) - f(z_0)g(z_0)| < (1 + |f(z_0)|)\frac{\varepsilon}{2(1+|f(z_0)|} + |g(z_0)|\frac{\varepsilon}{2|g(z_0)|}$$
$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$|f(z)g(z) - f(z_0)g(z_0)| < \varepsilon$ whenever $0 < |z - z_0| < \delta$
$\therefore \lim_{z\to z_0} f(z)g(z) = f(z_0)g(z_0)$
That is $f(z)g(z)$ is continuous at z_0

(iii) We get,

$$\left|\frac{1}{g(z)} - \frac{1}{g(z_0)}\right| = \frac{|g(z) - g(z_0)|}{|g(z_0)||g(z)|}$$

Since $\lim_{z\to z_0} g(z) = g(z_0) \neq 0$, so there exists $\delta_1 > 0$ such that $|g(z) - g(z_0)| < \frac{|g(z_0)|^2\varepsilon}{2}$for $|z - z_0| < \delta_1$
Again since $g(z)$ is continuous at z_0, then $\lim_{z\to z_0} g(z) = g(z_0)$. So there exists $\delta_2 > 0$ such that $|g(z) - g(z_0)| < \frac{|g(z_0)|}{2}$ whenever $|z - z_0| < \delta_2$.
But $g(z_0) = g(z_0) - g(z) + g(z)$

$$\therefore |g(z_0)| \le |g(z) - g(z_0)| + |g(z)|$$
$$< \frac{|g(z_0)|}{2} + |g(z)|$$

$or\ |g(z)| > \frac{1}{2}g(z_0)$

$or\ \frac{1}{|g(z)|} < \frac{2}{|g(z_0)|}$

Let $\delta = \min(\delta_1, \delta_2)$
So for $|z - z_0| < \delta$ we get,

$$\left|\frac{1}{g(z)} - \frac{1}{g(z_0)}\right| = \frac{|g(z) - g(z_0)|}{|g(z_0)|}.\frac{1}{|g(z)|}$$
$$\le \frac{|g(z_0)|^2\varepsilon}{|g(z_0)|^2}.2|g(z_0)| = \varepsilon$$

Or $\left|\frac{1}{g(z)} - \frac{1}{g(z_0)}\right| < \varepsilon$ whenever $0 < |z - z_0| < \delta$
Therefore $\lim_{z\to z_0} \frac{1}{g(z)} = \frac{1}{g(z_0)}$
That is $\frac{1}{g(z)}$ is continuous at z_0.
So by (ii), $f(z).\frac{1}{g(z)} = \frac{f(z)}{g(z)}$ is continuous at z_0.

Example: Show that $f(z) = z^2$ is uniformly continuous on |z|<1.

Solution: We must show that for any $\varepsilon > 0$ there exists a $\delta > 0$ (depending only on ε) such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$.

Here z_1 and z_2 are any two point in |z|<1.

Now, $|f(z_1) - f(z_2)| = |z_1^2 - z_2^2|$

$$= |z_1 + z_2||z_1 - z_2|$$
$$\le \{|z_1| + |z_2|\}|z_1 - z_2| \quad [\because\ |z_1| < 1,\ |z_2| < 1\ ]$$
$$< 2|z_1 - z_2|$$
$$< 2\delta \qquad \text{whenever } |z_1 - z_2| < \delta$$

Choose $\delta = \frac{\varepsilon}{2}$, we see that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$

Hence $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

Example: Show that $f(z) = \frac{1}{z}$ is not uniformly continuous on the region |z|<1.

Solution: If $f(z)$ is uniformly continuous in the region $|z| < 1$ then for any $\varepsilon > 0$ there exist a δ>0 (depending on ε) such that for any two points $z_1$and z_2 in the region, $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$.

Consider $\delta < 1$. Choose two points $z_1 = \delta_1$ and $z_2 = \frac{\delta}{1+\varepsilon}$.

Then $|z_1 - z_2| = \left|\delta - \frac{\delta}{1+\varepsilon}\right| = \frac{\delta\varepsilon}{1+\varepsilon} < \delta$

But, $|f(z_1) - f(z_2)| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$

$$= \left|\frac{1}{\delta} - \frac{1+\varepsilon}{\delta}\right|$$
$$= \frac{\varepsilon}{\delta} > \varepsilon \qquad [\because\ \delta < 1]$$

This is contradiction. Hence $f(z) = \frac{1}{z}$ is not uniformly continuous on the region |z|<1.

Differentiation

Definition: A function $f(z)$ is said to be differentiable at a point z_0 if $\lim\limits_{\Delta z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exist and we write

$f'(z_0) = \lim\limits_{\Delta z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

Here $f'(z_0)$ is called the derivative of $f(z)$ at z_0.

If we put $z = z_0 + \Delta z$

As $z \to z_0$ then $\Delta z \to 0$

Then $f'(z_0) = \lim\limits_{\Delta z \to z_0} \frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z}$

Theorem: A function $f(z)$ which is differentiable at a point is also continuous there. But the converse is not necessarily true.

Proof: Suppose $f(z)$ is differentiable at z_0. Then,

$f'(z_0) = \lim\limits_{\Delta z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exist and a finite number.

Now, $f(z) - f(z_0) = \frac{f(z)-f(z_0)}{z-z_0} \times (z - z_0)$

Taking limit on both sides as $z \to z_0$

$$\lim_{z\to z_0}\{f(z) - f(z_0)\} = \lim_{z\to z_0}\left\{\frac{f(z) - f(z_0)}{z - z_0} \times (z - z_0)\right\}$$

$$= \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \times \lim_{z \to z_0}(z - z_0)$$

$$= f'(z) \times 0 = 0$$

Hence, $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Thus, $f(z)$ is continuous at z_0.

To prove the converse, consider $f(z) = |z|$.

Here $f(z)$ is clearly continuous at $z = 0$.

But, $\lim\limits_{\Delta z \to 0} \frac{f(0+\Delta z)-f(0)}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{f(\Delta z)-f(0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|-|0|}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x+i\Delta y}$$

Suppose $\Delta z \to 0$ along $\Delta x = 0, \Delta y \to 0$

Then,

$$\text{limit} = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x+i\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{i\Delta y} = \frac{1}{i} = -i$$

Suppose $\Delta z \to 0$ along $\Delta x \to 0, \Delta y = 0$

Then,

$$\text{limit} = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x+i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Since the two limits are unequal, so $\lim\limits_{\Delta z \to 0} \frac{f(\Delta z)-f(0)}{\Delta z}$ does not exist or $f'(0)$ does not exist.

Definition: Let $f(z)$ be a function define on region R. Then $f(z)$ is said to be analytic in R if it is differentiable at every point of R.

Definition: A function $f(z)$ is called analytic at a point z_0 if there exists a neighbourhood of z_0 such that $f'(z)$ exists at each point of the neighbourhood.

Example: Show that $f(z) = |z|^2$ is differentiable at z=0 but not analytic there.

Proof: Here $f(z) = |z|^2$

Then, $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{|z|^2-0}{z}$

$$= \lim_{z\to 0} \frac{|z|^2}{z}$$

$$= \lim_{z\to 0} \frac{z\bar{z}}{z}$$

$$= \lim_{z\to 0} \bar{z} = 0$$

Hence f(z) is differentiable at z=0

Choose any point $z_0 \neq 0$, and consider the limit:

$$\lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(x_0+\Delta x)^2+(y_0+\Delta y)^2-(x_0^2+y_0^2)}{\Delta x+i\Delta y}$$

where, $\Delta z = \Delta x + i\Delta y$

$z_0 = x_0 + iy_0$

$$= \lim_{\Delta x\to 0, \Delta y\to 0} \frac{2x_0\Delta x+2y_0\Delta y+(\Delta x)^2+(\Delta y)^2}{\Delta x+i\Delta y}$$

Suppose $\Delta z \to 0$ along $\Delta x = 0, \Delta y \to 0$, then

$$\text{limit} = \lim_{\Delta x=0, \Delta y\to 0} \frac{2x_0\Delta x+2y_0\Delta y+(\Delta x)^2+(\Delta y)^2}{\Delta x+i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y_0\Delta y + (\Delta y)^2}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y_0 + \Delta y}{i} = \frac{2y_0}{i} = -2iy_0$$

Again, let $\Delta z \to 0$ along $\Delta x \to 0, \Delta y = 0$, then

$$\text{limit} \quad = \lim_{\Delta x \to 0, \Delta y = 0} \frac{2x_0\Delta x + 2y_0\Delta y + (\Delta x)^2 + (\Delta y)^2}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{2x_0\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0$$

Since the two limits are unequal, so $\lim\limits_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ does not exist.

Therefore $f(z)$ is not differentiable anywhere other than zero or origin.

Hence $f(z)$ is not analytic at $z = 0$.

Cauchy-Riemann equations (C-R equations): If $f(z) = u(x,y) + iv(x,y)$ defined in a region R, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called C-R equations.

Theorem: Necessary condition for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R is that it satisfies the C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at each point of R.

Proof: Since f(z) is analytic in R, so $f'(z)$ exists for each point z in R.

Hence $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \lim_{\Delta x \to 0, \Delta y \to 0} \frac{[\{u(x+\Delta x, y+\Delta y) - u(x,y)\} + i\{v(x+\Delta x, y+\Delta y) - v(x,y)\}]}{\Delta x + i\Delta y}$$

Must exist independent of manner in which Δz approaches zero (or $\Delta x, \Delta y$ approaches zero).

Consider the path $\Delta x = 0, \Delta y \to 0$, then

$$\text{limit} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i\{v(x, y+\Delta y) - v(x,y)\}}{i\Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Consider the path $\Delta y = 0, \Delta x \to 0$, then

$$\text{limit} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Since $f'(z)$ exists, above two limits must be equal. Hence

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Equating the real and imaginary part from both sides we get:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So C-R equations are satisfied.

Theorem: If $w = f(z) = u(x,y) + i\,v(x,y)$ satisfy the C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in a region R and partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in R, then $f(z)$ is analytic in R.

Proof: Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are supposed continuous, so

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y + \varepsilon_1\Delta x + \eta_1\Delta y$$

Where $\varepsilon_1 \to 0 \; as \; \Delta x \to 0$ and $\eta_1 \to 0 \; as \; \Delta y \to 0$

Similarly, since $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are supposed continuous, so

$$\Delta v = \left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y$$

$$= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2\Delta x + \eta_2\Delta y$$

Where $\varepsilon_2 \to 0 \; as \; \Delta x \to 0 \; and \; \eta_2 \to 0 \; as \; \Delta y \to 0$

Hence, $\Delta w = \Delta u + i\Delta v$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + (\varepsilon_1 + i\varepsilon_2)\Delta x + (\eta_1 + i\eta_2)\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

Where $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0 \; as \; \Delta x \to 0$

And $\eta = \eta_1 + i\eta_2 \to 0 \; as \; \Delta y \to 0$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$\therefore \Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y$$

i.e., $\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z + \varepsilon\Delta x + \eta\Delta y$ | where $\Delta z = \Delta x + i\Delta y$

$\therefore \frac{\Delta w}{\Delta z} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \frac{\varepsilon\Delta x + \eta\Delta y}{\Delta z}$

Notice that, $\left|\frac{\varepsilon\Delta x}{\Delta z}\right| = |\varepsilon|\left|\frac{\Delta x}{\Delta z}\right| \leq |\varepsilon|$ as $|\Delta x| \leq |\Delta z|$, which tends to zero as $\Delta z \to 0$

i.e., $\lim_{\Delta z\to 0}\frac{\varepsilon\Delta x}{\Delta z} = 0$

similarly, $\lim_{\Delta z\to 0}\frac{\eta\Delta y}{\Delta z} = 0$

So, $\lim_{\Delta z\to 0}\frac{\varepsilon\Delta x+\eta\Delta y}{\Delta z} = 0$

Hence, $\lim\limits_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Therefore, $\frac{dw}{dz} = f'(z)$ exist at each point of the region R, and hence f(z) is analytic in R.

Note: $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Note: *If $f(z) = u + iv$ is analytic then*

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\partial y}$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\partial y}$

Therefore for analytic function $\frac{\partial^2 v}{\partial x\partial y} = \frac{\partial^2 v}{\partial y\partial x}$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called Laplace's equation.

Definition: A function which satisfy Laplace's equation is called harmonic function.

Note: *So u is called harmonic function, similarly v is called harmonic function. u is called harmonic conjugate of v and v is called harmonic conjugate of u.*

Example: Show that $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, at\ z \neq 0 \\ 0\ at\ z = 0 \end{cases}$ is continuous everywhere. Also show that the C-R equations are satisfied at origin, but is not analytic there.

Proof:

Part1:

Let $f(z) = u + iv$

Now for $z \neq 0$

$$f(z) = \frac{\boldsymbol{x^3(1+i) - y^3(1-i)}}{\boldsymbol{x^2+y^2}}$$

$$= \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}$$

$\therefore u(x,y) = \frac{x^3-y^3}{x^2+y^2}$ $\qquad$ where $(x,y) \neq (0,0)$

$= 0$ at (0,0)

$\therefore v(x,y) = \frac{x^3+y^3}{x^2+y^2}$ $\qquad$ where $(x,y) \neq (0,0)$

$= 0$ at (0,0)

It is clear that $x^3 - y^3, x^3 + y^3, x^2 + y^2$ are continuous everywhere.

So for $z \neq 0$ both u and v are continuous.

Hence $f(z)$ is continuous for all $z \neq 0$.

Now for $z = 0$, choose $\varepsilon > 0$, then

$|u(x,y) - 0| = \left|\frac{x^3 - y^3}{x^2 + y^2}\right| = \frac{|x-y||x^2+xy+y^2|}{|x^2+y^2|}$

$\leq \frac{|x-y|\,2|x^2+y^2|}{|x^2+y^2|}$

$= 2|x - y|$

$\leq 2(|x| + |y|) \leq 4\sqrt{x^2+y^2}$

$\because \frac{x^2+y^2}{2} \geq xy$

$\Rightarrow x^2 + y^2 \geq xy$

$\because |x| + |y| \leq \sqrt{2}|x + iy|$

$= \sqrt{2}\sqrt{x^2+y^2}$

$= 2\sqrt{x^2+y^2}$

Consider, $\delta = \frac{\varepsilon}{4}$, then

$|u(x,y) - 0| < 4.\frac{\varepsilon}{4}$ whenever $\sqrt{x^2+y^2} < \delta$

i.e., $|u(x,y) - 0| < \varepsilon$ whenever $|(x,y) \to (0,0)| < \delta$

Hence, $\lim_{(x,y)\to(0,0)} u(x,y) = 0$

i.e., $u(x,y)$ is continuous at (0,0)

Similarly $v(x,y)$ is continuous at $(0,0)$

Hence $f(z)$ is continuous at $z = 0$

There $f(z)$ is continuous everywhere.

Part 2:

Now we get,

$\left.\frac{\partial u}{\partial x}\right|_{0,0} = \lim_{\Delta x\to 0}\frac{u(\Delta x,0)-u(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(\Delta x)^3}{(\Delta x)^2\Delta x} = 1$

$\left.\frac{\partial u}{\partial y}\right|_{0,0} = \lim_{\Delta x\to 0}\frac{u(0,\Delta y)-u(0,0)}{\Delta y} = \lim_{\Delta x\to 0}\frac{-(\Delta y)^3}{(\Delta y)^2\Delta y} = -1$

$\left.\frac{\partial v}{\partial x}\right|_{0,0} = \lim_{\Delta x\to 0}\frac{v(\Delta x,0)-v(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(\Delta x)^3}{(\Delta x)^2\Delta x} = 1$

$\left.\frac{\partial v}{\partial y}\right|_{0,0} = \lim_{\Delta x\to 0}\frac{v(0,\Delta y)-v(0,0)}{\Delta y} = \lim_{\Delta x\to 0}\frac{(\Delta y)^3}{(\Delta y)^2\Delta y} = -1$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Therefore C-R equations are satisfied at (0, 0)

Consider $\lim_{z\to 0}\frac{f(z)-f(0)}{z}$

$= \lim_{x\to 0,y\to 0}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$

Let $z\to 0$ along x=0, $y\to 0$. Then

$$limit = \lim_{y\to 0}\frac{-y^3(1-i)}{y^2(iy)} = \lim_{y\to 0}\frac{i-1}{i} = \frac{i-1}{i} = 1+i$$

Again let $z\to 0$ along y=x. Then

$$limit = \lim_{x\to 0}\frac{x^3(1+i)-x^3(1-i)}{2x^2(x+ix)} = \lim_{x\to 0}\frac{1+i-1+i}{2(1+i)} = \frac{i}{1+i}$$

Since the two limits are unequal so $f'(0)$ does not exist.

So $f(z)$ is not analytic at $z=0$.

Example: Show that $f(z) = \begin{cases}\frac{x^2y^6(x+iy)}{x^4+y^{12}} & when\ z\neq 0\\ 0 & when\ z=0\end{cases}$ is not analytic at origin (0, 0), but C-R equations are satisfied there.

Solution: Consider $\lim_{z\to 0}\frac{f(z)-f(0)}{z}$

$= \lim_{x\to 0,y\to 0}\frac{y^6x^2(x+iy)}{(x^4+y^{12})(x+iy)}$

Consider the path *y=mx*, then

$$limit = \lim_{x\to 0}\frac{m^6x^8}{x^4+m^{12}x^{12}} = \lim_{x\to 0}\frac{m^6x^4}{1+m^{12}x^8} = 0$$

Again consider the path $x^2=y^6$. Then

$$limit = \lim_{y\to 0}\frac{y^{12}}{y^{12}+y^{12}} = \frac{1}{2}$$

Since the two limits are unequal so $f'(0)$does not exist and hence $f(z)$ is not analytic at origin $(0,0)$.

Let $f(z) = u+iv$. Then

$u = \begin{cases}\frac{x^3y^6}{x^4+y^{12}}\ when\ (x,y)\neq 0\\ 0\ at\ (x,y)=0\end{cases}$ $v = \begin{cases}\frac{x^2y^7}{x^4+y^{12}}\ when\ (x,y)\neq 0\\ 0\ at\ (x,y)=0\end{cases}$

Now

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{\Delta x\to 0}\frac{u(x,0)-u(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{0-0}{\Delta x} = 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{\Delta y\to 0}\frac{u(0,y)-u(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{0-0}{\Delta y} = 0$$

Similarly we can show that $\left.\frac{\partial v}{\partial x}\right|_{0,0} = \left.\frac{\partial v}{\partial y}\right|_{0,0} = 0$

Hence, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i.e., C-R equations are satisfied at $(0,0)$

Example: Show that $f(z) = 2x+ixy^2$is not analytic at anywhere.

Solution: Let $f(z) = u+iv$, then

$u=2x, v=xy^2$. So,

$$\frac{\partial u}{\partial x} = 2, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = y^2, \frac{\partial v}{\partial y} = 2xy$$

Hence $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ when $y\neq 0$

i.e., C-R equations are not satisfied for any $z\neq 0$.

More over at $z=0$, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$\therefore f(z)$ is not analytic anywhere.

Example: Let $u = 3x^2y+2x^2-y^3-2y^2$. Show that u is harmonic and find its harmonic conjugate *v*. Then find$f(z) = u+iv$ in terms of z.

Solution: Here $u = 3x^2y+2x^2-y^3-2y^2$

$$\frac{\partial u}{\partial x} = 6xy+4x$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4$$

$$\frac{\partial^2 u}{\partial y^2} = -6y^2 - 4$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i.e., u is a harmonic function.

Now by C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\therefore \frac{\partial v}{\partial y} = 6xy + 4x$$

$$v = 3xy^2 + 4xy + F(x)$$

$$\frac{\partial v}{\partial x} = 3y^2 + 4y + F'(x)$$

$\Rightarrow -3x^2 + 3y^2 + 4y = 3y^2 + 4y + F'(x)$

$\Rightarrow F'(x) = -3x^2$

$\Rightarrow F(x) = -x^3 + C$

$\therefore$ $v = 3xy^2 + 4xy - x^3 + C$, the harmonic conjugate of u.

Now, $f(z) = u + iv$

$= 3xy^2 + 2x^2 - y^3 - 2y^2 + i[3xy^2 + 4xy - x^3 + C]$

$= 2\{x^2 + 2xiy + (iy)^2\} - i\{x^3 + 3x(iy)^2 + 3x^2 iy + (iy)^3\}$

$= 2(x + iy)^2 - i(x + iy)^3 + iC$

$= 2z^2 - iz^3 + iC$

Example: Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Find the analytic $f(z) = u + iv$ in terms of z.

Solution: Here $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \varphi_1(x, y)\ say$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6$$

$$\frac{\partial u}{\partial y} = -6xy - 6y = \varphi_2(x, y)\ say$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0$

i.e., u is a harmonic function.

Now by Milne's method.

$f'(z) = \varphi_1(z, 0) - i\varphi_2(z, 0) = 3z^2 + 6z - i(0) = 3z^2 + 6z$

$\therefore f(z) = z^3 + 3z^2 + C$

Example: Show that an analytic function of constant modulas is a constant.

Proof: $f(z) = u + iv$

$\therefore |f(z)| = \sqrt{u^2 + v^2} = \sqrt{k} \neq 0(constant)$ ________________________________ (1)

Differentiating (1) with respect to x we get,

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$ ______________________________________ (2)

Differentiating (1) with respect to y we get,

$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \Rightarrow -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ _______________________________________ (3) [By C-R equations]

Squaring (2) and (3) and adding we have,

$$(u^2 + v^2)\left(\frac{\partial u}{\partial x}\right)^2 + (u^2 + v^2)\left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$$\Rightarrow (u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad [\because (u^2 + v^2) \neq 0]$$

$$\Rightarrow |f'(z)|^2 = 0$$

i.e., $f'(z) = 0$

$$\therefore f(z) = constant$$

Example: Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$when f(z) is analytic functions.

Solution: Let $f(z) = u + iv$. Then,

$$|f(z)|^2 = u^2 + v^2$$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 v}{\partial x^2}$ ___________________(1)

Similarly,

$$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2v\frac{\partial^2 v}{\partial y^2}$$

$= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2v\frac{\partial^2 v}{\partial y^2}$___________________(2) [By C-R equations]

From (1) and (2) we get,

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$ _______________(3)

As both u and v are harmonic so,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

So from (3) we get,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 4|f'(z)|$$

General Formula: Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}$ where $z = x + iy$ provided $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$.

Solution: We have $z = x + iy, \bar{z} = x - iy$

$$\therefore x = \frac{1}{2}(z + \bar{z}) \qquad y = \frac{1}{2i}(z - \bar{z}) = -\frac{i}{2}(z - \bar{z})$$

$$\therefore \frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$$

$$\frac{\partial y}{\partial z} = -\frac{i}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Now $\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$

$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

And, $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$\therefore \frac{\partial^2}{\partial z\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}$ **[Proved]**

Example: Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^k = k^2|f(z)|^{k-2}|f(z)|^2$ where f(z) is analytic function.

Solution: We get,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^k = 4\frac{\partial^2}{\partial z\partial \bar{z}}\{|f(z)|^2\}^{\frac{k}{2}}$$

$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\{f(z)\overline{f(z)}\}^{\frac{k}{2}}$

$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\{f(z)^{\frac{k}{2}}f(\bar{z})^{\frac{k}{2}}\}$ $\quad \because \overline{f(z)} = f(\bar{z})$ as $f(z)$ is analytic

$= 4\frac{\partial}{\partial z}\{f(z)^{\frac{k}{2}}\frac{k}{2}f(\bar{z})^{\frac{k}{2}-1}f'(\bar{z})\}$

$= 4[\frac{k}{2}f(z)^{\frac{k}{2}-1}f'(z)\frac{k}{2}f'(\bar{z})^{\frac{k}{2}-1}f'(\bar{z})]$

$= k^2f(z)^{\frac{k-2}{2}}f(\bar{z})^{\frac{k-2}{2}}f'(z)f'(\bar{z})$

$= k^2\{f(z)\overline{f(z)}\}^{\frac{k-2}{2}}|f'(z)|^2$

$= k^2|f(z)|^{k-2}|f'(z)|^2$ **[Proved]**

Definition: A Point at which a function $f(z)$ is not analytic or fails to be analytic is called a singular point or singularity of $f(z)$.

Definition: A singularity z_0 of $f(z)$ is called a pole of order n if $\lim_{z\to z_0}(z - z_0)^n f(z)$ exist and $\neq 0$.

For example,

If $f(z) = \frac{z^2+3}{(z-2)^3(z+3)}$ then

$z = 2$ is a pole of order 3 of $f(z)$.

z=-3 is pole of order 1 of $f(z)$.

Definition: A singularity z_0 of $f(z)$ is said to be isolated singularity if there exists a neighborhood N of z_0 which contains no singularity other than z_0.

Definition: A singularity z_0 of f(z) is called a removable singularity if $\lim_{z\to z_0} f(z)$ exists.

For example,

$$f(z) = \frac{\sin z}{z}$$

$z = 0$ is a singular point.

But $\lim_{z\to 0}\frac{\sin z}{z} = 1$

Therefore $z = 0$ is a removable singularity.

Definition: A singularity z_0 of $f(z)$ which is neither a pole nor an isolated singularity is called essential singularity.

For example, $f(z) = e^{\frac{1}{z-2}}$ has an essential singularity at $z = 2$.

Definition: A point z_0 is called a singularity at infinity of $f(z)$ if z_0 is a pole of order m of $f\left(\frac{1}{z}\right)$.

Definition: A function f(z) has a singularity at infinity if $f\left(\frac{1}{w}\right)$ has a singularity at $w = 0$.

For example,

$$f(z) = z^3 \therefore f\left(\frac{1}{w}\right) = \frac{1}{w^3}$$

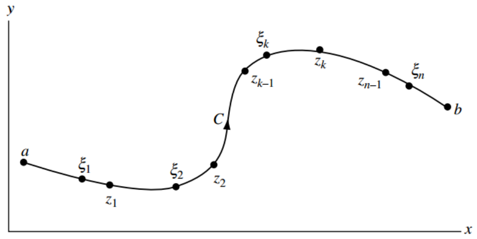
Here $w = 0$ is a pole of order 3 of $f\left(\frac{1}{w}\right)$.

Hence $z = \infty$ is a pole of order 3 of $f(z)$.

Complex Integration

Definition: let $f(z)$ be continuous on a curve C of finite length. let us divide the curve by the points, $a = z_0,\ z_1,\ z_2,\ \ldots\ldots,\ z_n = b$

Now for the sum, $S_n = \sum_{k=1}^{n} f(\xi_k)\Delta z_k$



where $\Delta z_k = z_k - z_{k-1}$ and ξ_k is any point on the curve between z_{k-1} and z_k.
Let us increase the number of sub division in such a way that the largest of the chord length $|\Delta z_k|$ approaches to zero . Then S_n approaches to a value which is known as complex line integral of $f(z)$ along C denoted by $\int_c f(z)dz$ or $\int_a^b f(z)dz$. I.e., $\int_c f(z)dz = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right)$

property of complex integration:

1. $\int_c (f(z) + g(z))dz = \int_c f(z)dz + \int_c g(z)dz$
2. $\int_c kf(z)dz = k\int_c (z)dz$
3. $\int_a^b (z)dz = -\int_b^a f(z)dz$
4. $\int_a^b f(z)dz = \int_a^m f(z)dz + \int_m^a f(z)dz$
5. $|\int_c f(z)dz| \le$ ML where k is the length of the curve c and $|f(z)| \le M \quad z \epsilon C$

Proof of (5) : by definition $\int_c f(z)dz = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right)$

Now, $\left|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right| \le \sum_{k=1}^{n}\left|f(\xi_k)\right| |\Delta z_k|$

$$\le M\sum_{k=1}^{n}|\Delta z_k|$$
$$\le ML$$

As $\sum_{k=1}^{n}|\Delta Z_k|$ is the sum of all chord length which must be less than the length of the curve. Taking limit on both side as n tends to infinity we have by definition

$$\left|\int_c f(z)dz\right| \le ML$$

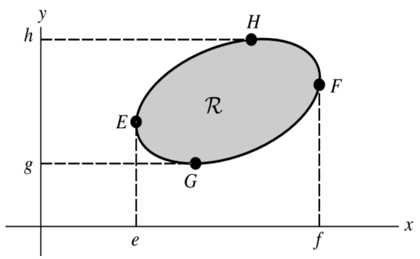
Definition: A closed curve which does not intersect itself is called a simple closed curve.
Definition: A region R is called simply connected region if every simple closed curve in R can be shrank to a point without leaving R . Otherwise R is called multiply connected region.

Greens Theorem: If $P(x,y)$ and $Q(x,y)$ are continuous and have continuous partial derivatives inside and on a simple closed curve C Then $\int_c Pdx + Qdy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy$ where R is the region bounded by C.
The theorem is valid for both simply and multiply-connected regions.

Example : Prove Green's theorem in the plane if C is a simple closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points

Proof:

Let the equation of the curve EGF and EHF be $y = Y_1(x)$ and $y = Y_2(x)$ respectively then,



$$\iint_R \left(\frac{\delta p}{\delta y} dy\right) dx = \int_{x=e}^{f} \left[\int_{y=Y_1}^{Y_2} \frac{\delta p}{\delta y} dy\right] dx$$
$$= \int_{x=e}^{f} P(x,y)\,\Big|_{y=Y_1(x)}^{Y_2(x)} dx$$
$$= \int_e^f [P(x,Y_2) - P(x,Y_1)]dx$$
$$= \int_e^f P(x,Y_2)dx - \int_e^f P(x,Y_1)dx$$
$$= -\int_e^f P(x,Y_1)dx - \int_f^e P(x,Y_2)dx$$
$$= -\oint_c Pdx$$

Then, $\oint_c Pdx = -\iint_R \frac{\delta p}{\delta y} dxdy$ ------- (1)

Now let the equation of the curve GEH and GFH be $x = X_1(y)$ and $x = X_2(y)$ respectively

Thus, $\iint_R \frac{\delta Q}{\delta x} dxdy = \int_{y=g}^{h} \left[\int_{x_1 y}^{x_2 y} \frac{\delta Q}{\delta x} dx\right] dy$

$$= \int_g^h [Q(x_2,y) - Q(x_1,y)]dy$$
$$= \int_g^h Q(x_2,y)dy + \int_g^h Q(x_1,y)dy$$
$$= \oint_c Qdy$$

Then $\oint_c Qdy = \iint \frac{\delta Q}{\delta x} dxdy$ ------------------ (2)

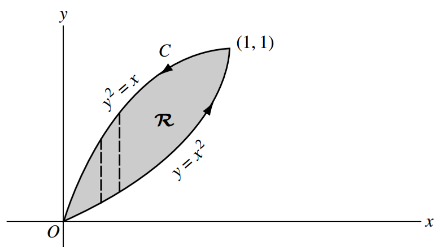
Adding (1) ,(2) we get ,

$$\oint_c pdx + Qdy = \iint_R \left(\frac{\delta Q}{\delta y} - \frac{\delta p}{\delta y}\right) dxdy$$

Example : Verify Greens theorem in the plane for

$$\oint_c (2xy - x^2)dx + (x + y^2)dy$$

Where C is the closed curve in the region bounded by $y = x^2$ and $x = y^2$



Solution: Here $P = 2xy - x^2$ and $Q = x + y^2$. The plane curves $y = x^2$ and $x = y^2$ intersect at (0,0) and (1,1). The positive direction of the curve C is as shown in the figure.

Along $y = x^2$, the line integral

$I_1 = \int_{x=0}^{1} \{(2x)(x^2) - x^2\}dx + \{x + (x^2)^2\}d(x^2)$

$= \int_0^1 (2x^3 + x^2 + 52)dx = \frac{7}{6}$

Along $y^2 = x$, the line integral

$I_2 = \int_{y=1}^{0} \{2(y^2)(y) - (y^2)^2\}d(y^2) + \{y^2 + y^2\}dy$

$= \int_1^0 (4y^4 - 2y^5 + 2y^2)dy = -\frac{17}{15}$

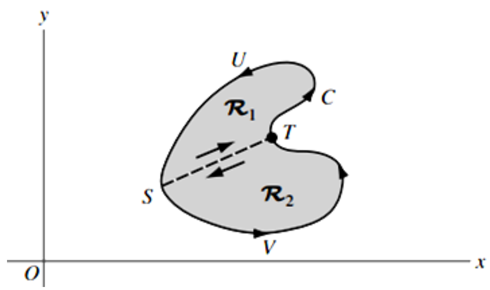
So the Line integral $= \frac{7}{6} - \frac{17}{15} = \frac{1}{30}$. On the other hand

$$\iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dxdy = \iint_R \left(\frac{\delta}{\delta x}(x + y^2) - \frac{\delta}{\delta y}(2xy - x^2)\right) dxdy$$
$$= \iint_R \left(\frac{\delta}{\delta x}(1 - 2x)dxdy\right.$$
$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1 - 2x)dydx$$
$$= \int_{x=0}^{1} (y - 2xy)\Big|_{y=x^2}^{\sqrt{x}} \, dx$$
$$= \int_0^1 (x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^2 + 2x^3)dx$$
$$= \frac{1}{30}$$

Hence the Green's theorem is verified.

Example : Extend the proof of Green's theorem in the plane to the curve C for which lines parallel to the coordinate axes may cut C in more than two points

Solution : Consider a simple closed curve C such as shown in the figure in which lines parallel to the axes may meet C in more than two points. By constructing line ST, the region is divided into two regions R_1 and R_2 whose boundaries are STUS and SVTS respectively which are of the type that any straight line parallel to the coordinate axes cuts C_1 and C_2 atmost two points and for which Green's theorem applies, i.e.



$$\oint_{SSUT} pdx + Qdy = \iint_{R_1} \left(\frac{\delta Q}{\delta y} - \frac{\delta p}{\delta y}\right) dxdy \ldots\ldots\ldots\ldots(1)$$

$$\oint_{SVTS} pdx + Qdy = \iint_{R_2} \left(\frac{\delta Q}{\delta y} - \frac{\delta p}{\delta y}\right) dxdy \ldots\ldots\ldots\ldots(2)$$

Adding the left-hand sides of (1) and (2), we have, omitting the integrand $Pdx + Qdy$ in each case.

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{SVS} + \int_{SVT} = \int_{SUSVT}$$

using the fact that using the fact that $\int_{ST} = -\int_{TS}$

Adding the right-hand sides of (1) and (2), omitting the integrand,

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

Then

$$\oint_c pdx + Qdy = \iint_R \left(\frac{\delta Q}{\delta y} - \frac{\delta p}{\delta y}\right) dxdy$$

and the theorem is proved.

NB: We have proved Green's theorem for the simply-connected region bounded by the simple closed curve C. For more complicated regions, it may be necessary to construct more lines, such as ST, to establish the theorem.

Green's theorem is also true for multiply-connected regions.

Theorem (Cauchy's theorem) : If $f(x)$ is analytic inside and on a simple closed curve C and if $f'(z)$ is continuous Then

$$\oint_c f(z)dz = 0$$

Proof. Let $f(z) = u + iv$

Then $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$

$= \frac{\delta u}{\delta y} - i\frac{\delta v}{\delta y}$ [by C-R equation]

Here $u, v, \frac{\delta u}{\delta x}, \frac{\delta v}{\delta x}, \frac{\delta u}{\delta y}, \frac{\delta v}{\delta y}$ were all continuous

hence $\oint_c f(z)dz = \oint_c (u + iv)(dx + idy)$

$= \oint_c (udx - vdy) + i\oint_c (vdx + udy)$

$= \iint_R \left(-\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right) dxdy + i\iint_R \left(\frac{\delta u}{\delta x} - \frac{\delta v}{\delta y}\right) dxdy$

$= 0$ [by C-R equations]

Where R is the region bounded by C

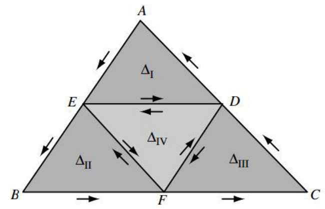
Theorem : (Cauchy's fundamental Theorem) :

If $f(z)$ is analytic inside and on a simple closed curve C then $\oint_c f(z)dz = 0$.

Or,

If $f(z)dz$ is analytic in a region and on the boundary bounded by a simple closed curve C Then $\oint_c f(z)dz = 0$.

Theorem : Prove Cauchy theorem for a triangle **.**



Proof : We know that if C is a simple closed curve then (i) $\oint_c dz = 0$ and (ii) $\oint_c zdz = 0$

Let $f(z)$ is analytic inside and on triangle ABC denoted by Δ. We need to prove that, $\oint_\Delta f(z)dz = 0$. Let D, E, and F are the middle points of BC , CA , AB Respectively. Join DE , EF , and FD then we obtain four triangle $\Delta_i, \Delta_{ii}, \Delta_{iii}$ and Δ_{iv}.

Now ,

$\oint_\Delta f(z)dz = \oint_{EAF} f(z)dz + \oint_{FBD} f(z)dz + \oint_{DCE} f(z)dz$

$= (\int_{EAF} + \int_{FE}) + (\int_{FBD} + \int_{DF}) + (\int_{DEF} + \int_{ED}) + (\int_{EF} + \int_{FD} + \int_{DE})$

$[\because \int_{FE} = -\int_{EF}]$

$= \int_{\Delta_i} + \int_{\Delta_{ii}} + \int_{\Delta_{iii}} + \int_{\Delta_{iv}}$

Hence , $\left|\int_\Delta f(z)dz\right| \le \left|\int_{\Delta_i} f(z)dz\right| + \left|\int_{\Delta_{ii}} f(z)dz\right| + \left|\int_{\Delta_{iii}} f(z)dz\right| + \left|\int_{\Delta_{iv}} f(z)dz\right|$

Let Δ_i represents that triangle above four so that $\left|\int_{\Delta_i} f(z)dz\right|$ is the largest (if these are more triangles with these property then we can choose any one of them Δ_i)

Thus , $\left|\int_\Delta f(z)dz\right| \le 4\left|\int_{\Delta_1} f(z)dz\right|$

Now joining the midpoints of the triangle Δ_1 *and* proceeding as above we obtain a triangle Δ_2

So that $\left|\int_\Delta f(z)dz\right| \le 4^2 \left|\int_{\Delta_2} f(z)dz\right|$

Proceeding similarly we obtain a sequence of triangle $\{\Delta_n\}$

Such that , $\Delta > \Delta_1 > \Delta_2 >$………..

And $\left|\int_\Delta f(z)dz\right| \le 4^n \left|\int_{\Delta_n} f(z)dz\right|$

Then there exist a point z_o such that $z_o \in \Delta_k$ for each $k = 1,2, \ldots\ldots n$.

We know that $f'(z) = \lim_{z \to z_0} \frac{f(Z) - F(Z_0)}{Z - Z_0}$

Therefore as $f(z)$ is analytic at z_o then for any $\varepsilon > 0$ there exist $\delta > 0$ such that

$f(z) = f(z_o) + f'(z_0)(z - z_0) + \eta(z - z_0)$; where $|\eta| < \varepsilon$ *whenever* $|z - z_0| < \delta$

$\int_{\Delta_n} f(z)dz = f(z_0)\int_{\Delta_n} dz + f'(z_0)\int_{\Delta_n}(z - z_0)dz + \int_{\Delta_n}\eta(z - z_0)dz$

$= \int_{\Delta_n} \eta(z - z_0)dz$; by (i) and (ii)

Now for any z on Δ_n $|z - z_0| < \frac{p}{2^n}$; where p is the perimeter of Δ.

Now $|\eta(z - z_0)| = |\eta||(z - z_0)| < \varepsilon \frac{p}{2^n}$.

Choose n sufficiently large so that $\frac{p}{2^n} < \delta$.

Then $\left|\int_{\Delta n} \eta\eta(z - z_0)dz\right| < \varepsilon\frac{p}{2^n}\frac{p}{2^n}$; when ever $|z - z_0| < \frac{p}{2^n} < \delta$

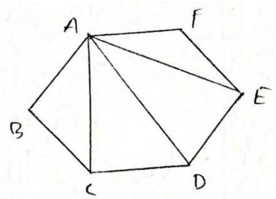
Thus $\left|\int_{\Delta_n} f(z)dz\right| \le \frac{\varepsilon p^2}{4^n}$; when ever $|z - z_0| < \frac{p}{2^n} < \delta$

Therefore $\left|\int_\Delta f(z)dz\right| \le 4^n \left|\int_{\Delta_n} f(z)dz\right| \le \varepsilon p^2$

Since ε *is* arbitrary so $\left|\int_\Delta f(z)dz\right| = 0$ i.e. $\int_\Delta f(z)dz = 0$

Ex. Prove Cauchy's theorem for a polygon**.**

Proof : Let ABCDEF be a polygon. Suppose $f(z)$ is analytic inside and on the polygon .we need to prove that $\int_{ABCDEF} f(z)dz = 0$. Join AC , AD , AE then

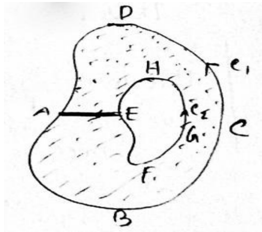


$$\int_{ABCDEF} f(z)dz = \int_{ABC} + \int_{CD} + \int_{DE} + \int_{EFA}$$
$$= (\int_{ABC} + \int_{CA}) + (\int_{CD} + \int_{DA} + \int_{AC}) + (\int_{DE} + \int_{EF} + \int_{AC})$$
$$+ (\int_{EFA} + \int_{AE})$$
$$= \int_{ABCA} + \int_{CDAC} + \int_{DEAD} + \int_{EFAE} = 0 + 0 + 0 + 0 = 0 .$$

[As the Cauchy's theorem holds for triangle]

Ex. If $f(z)$ is a analytic inside and on the region bounded by C_1 *and* C_2 as shown by the figure. Then $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$

Proof : Let ABCDA represent the curve C_1 *and EFGH represent* C_2. Construct a cross cut AE then AEHGFEABCDA becomes a closed curve. Moreover $f(z)$ is analytic inside and on the region. Then by Cauchy's theorem



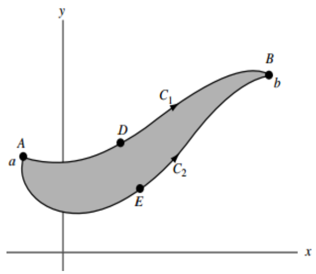
$$\int_{\text{AEHGFEABCDA}} f(z)dz = 0$$
$$\Rightarrow \int_{AE} + \int_{EHGFE} + \int_{EA} + \int_{ABCDA} = 0$$
$$\Rightarrow \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$
$$\therefore \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Ex *:* Suppose $f(z)$ is analytic in a simply-connected region R. Prove that $\int_a^b f(z)dz$ is independent of the path in R joining any two points a and b in R.

Proof: Consider the two paths $C_1 = ADB$ and $C_2 = AEB$ from a to b in R as shown by the figure. Now we have to show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Clearly $AEBDA$ is a simple closed curve in R Then by Cauchy's theorem



$$\int_{\text{AEBDA}} f(z)dz = 0$$

or, $\int_{\text{AEB}} f(z)dz + \int_{\text{BDA}} f(z)dz = 0$

or, $\int_{\text{AEB}} f(z)dz - \int_{\text{ADB}} f(z)dz = 0$

or, $\int_{C_2} f(z)dz - \int_{C_1} f(z)dz = 0$

or, $f(z)dz = \int_{C_2} f(z)dz$

Cauchy's Integral Formulas and Related Theorems

If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C. Then,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Proof: Construct a circle Γ inside C with centre at a and radius r. Then $\frac{f(z)}{z-a}$ is analytic inside the region bounded by Γ and C and also on the boundary. Then by cauchys theorem for simple connected region we get

$$\oint_C \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{z-a}dz \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now on Γ

$$|z-a| = r$$

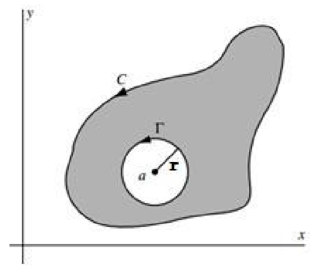
i.e.,

$$z - a = re^{i\theta}$$

i.e.,

$$z = a + re^{i\theta} \Rightarrow dz = i\, re^{i\theta} d\theta \qquad \text{where} \quad 0 \le \theta \le 2\pi$$

Hence



$$\oint_\Gamma \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta$$

i.e.,

$$\oint_C \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a+ re^{i\theta})d\theta \qquad [by (i)]$$

Taking limit on both sides as $r \to 0$ we have

$$\oint_C \frac{f(z)}{z-a}dz = i\lim_{r\to 0}\int_0^{2\pi} f(a+re^{i\theta})d\theta$$

$$= i\int_0^{2\pi} \lim_{r\to 0} f(a+re^{i\theta})d\theta \qquad [\text{as } f(z) \text{ is analytic}]$$

$$= i\int_0^{2\pi} \lim_{r\to 0} f(a)d\theta \qquad [\text{as } f(z) \text{ continous at } a]$$

$$= 2\pi i\, f(a)$$

Hence

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Example. Show that $\oint_C \frac{dz}{z-a}=\begin{cases}0 & \text{if } a \text{ is outside } C\\ 2\pi i & \text{if } a \text{ is inside } C\end{cases}$

Where C a simple closed curve.

Solution: If a is outside C then $\frac{1}{z-a}$ is analytic inside and on C. So by cauchys theorem

$$\oint_C \frac{dz}{z-a}=0 \qquad [proved]$$

If a is inside C, then put $f(z)=1$

Then by cauchys integral formula

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

i. e.,

$$1=\frac{1}{2\pi i}\oint_C \frac{dz}{z-a}$$

i. e.,

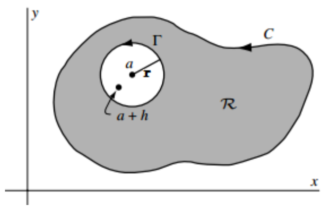
$$\oint_C \frac{dz}{z-a}=2\pi i$$

Theorem: If $f(z)$ is analytic inside and on a simple closed curve C and a is any pint inside C, Then

$$f'(a)=\frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz$$

Proof: Choose a point $a+h$ inside C. Then any cauchys integral formula

$$f(a+h)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a-h}dz$$



And

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Now

$$\frac{f(a+h)-f(a)}{h}=\frac{1}{h2\pi i}\oint_C \left(\frac{1}{z-a-h}-\frac{1}{z-a}\right)f(z)dz$$

$$=\frac{1}{h2\pi i}\oint_C \frac{h}{(z-a-h)(z-a)}f(z)\,dz$$

$$=\frac{1}{2\pi i}\oint_C \frac{(z-a)}{(z-a-h)(z-a)^2}f(z)dz$$

$$=\frac{1}{2\pi i}\oint_C \frac{(z-a-h)+h}{(z-a-h)(z-a)^2}f(z)dz$$

$$=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz+\frac{1}{2\pi i}\oint_C \frac{h\,f(z)}{(z-a-h)(z-a)^2}dz$$

Construct a circle Γ inside C with centre at a and radius r. Choose $|h|$ small enough so that $a+h$ lies inside Γ and $|h|<\frac{r}{2}$. Since $f(z)$ is analytic inside and on C. So $|f(z)|<M$ for some +ve integer M. Now on Γ,

$$|z-a|=r$$

$\therefore$

$$|z-a-h|\geq|z-a|-|h|>r-\frac{r}{2}=\frac{r}{2}$$

Hence

$$\left|\frac{1}{2\pi i}\oint_C \frac{h\ f(z)\ dz}{(z-a-h)(z-a)^2}\right|=\left|\frac{1}{2\pi i}\oint_\Gamma \frac{h\ f(z)\,dz}{(z-a-h)(z-a)^2}\right|<\frac{1}{2\pi}\frac{|h|M}{\frac{r}{2}\ r^2}\ 2\pi r$$

$$=\frac{2M|h|}{r^2}\to 0 \quad as\ h\to 0$$

Hence

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\frac{1}{2\pi i}\oint_C \frac{f(z)\ dz}{(z-a)^2}+0$$

i.e.,

$$f'(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)\ dz}{(z-a)^2}$$

(Proved)

Cauchys general integral formula: If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C. Then

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}} \; dz$$

Proof: We use the method of induction. The formula holds for $n = 0$ $[cauchys\ integral\ formula]$

By the preceding theorem this formula holds for $n = 1$ (একটা অন্তত দেখাতে হবে)

Suppose the formula holds for formula holds for $n = m - 1$. Then

$$f^{m-1}(a) = \frac{(m-a)!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^m} dz$$

Choose any $a + h$ inside C then

$$\frac{f^{m-1}(a+h) - f^{m-1}(a)}{h} = \frac{(m-1)!}{h2\pi i}\oint_C \left\{\frac{1}{(z-a-h)^m} - \frac{1}{(z-a)^m}\right\} f(z)\, dz$$

$$= \frac{(m-1)!}{h2\pi i}\oint_C \left\{\frac{1}{(z-a)^m(1-\frac{h}{z-a})^m} - \frac{1}{(z-a)^m}\right\} f(z)\, dz$$

$$= \frac{(m-1)!}{h2\pi i}\oint_C \left\{\frac{1}{(z-a)^m}\left[\left(1-\frac{h}{z-a}\right)\right]^{-m} - 1\right\} f(z)\, dz$$

$$= \frac{(m-1)!}{h2\pi i}\oint_C \frac{1}{(z-a)^m}\left[1 + \frac{mh}{z-a} + \frac{m(m+1)h^2}{2!\,(z-a)^2} + \cdots + higher\ power\ of\ h - 1\right] f(z)dz$$

$$= \frac{(m-1)!}{h2\pi i}\oint_C \left\{\frac{m}{(z-a)^{m+1}} + \frac{m(m+1)\,h}{2!\,(z-a)^{m+2}} + \cdots \quad + higher\ power\ of\ h\right\} f(z)dz$$

Taking limit on the both side as h→0 we get

$$\lim_{h\to 0}\frac{f^{m-1}(a+h) - f^{m-1}(a)}{h}$$
$$= \lim_{h\to 0}\frac{(m-1)!}{2\pi i}\oint_C \left\{\frac{m}{(z-a)^{m+1}} + \frac{m(m+1)h}{2!\,(z-a)^{m+2}} + \cdots higher\ power\ of\ h\right\} f(z)dz$$

i.e.,

$$f^m(a) = \frac{(m-1)!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{m+1}} \, f(z)dz$$

i.e.,

$$f^m(a) = \frac{m!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{m+1}} dz$$

i.e., the formula holds for n=m. Hence by the method of inductions the formula holds for all +ve integer n.

Cauchys inequality: If $f(z)$ is analytic inside and on a circle C with centre at a and radius r and if $|f(z)| < M$ on C. then

$$|f^n| \le \frac{Mn!}{r^n}$$

Proof: By Cauchys general integral formula

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n-1}} \; dz$$

Hence

$$|f^n(a)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}} \; dz\right| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r$$

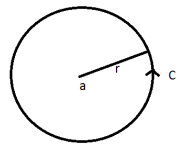
[∴ length of $C = 2\pi r$ and on $C, (z-a) = r, |f(z)| < M$]

i.e.,

$$|f^n(a)| \le \tfrac{n!M}{r^n} \qquad [Proved]$$

Gauss mean value theorem: If $f(z)$ is analytic inside and on a circle C with centre at a and radius r, then

$$f(a) = \tfrac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta$$



Proof: By Cauchys integral formula

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a} dz$$

On C,

$$z - a = re^{i\theta}$$

i. e.,

$$dz = ire^{i\theta} \, d\theta$$

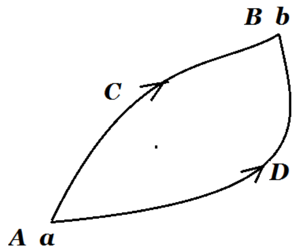
∴

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f\left(a+re^{i\theta}\right)ire^{i\theta}d\theta}{re^{i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f\left(a+re^{i\theta}\right)d\theta$$

Example. If in a region R, $\oint_C f(z)dz = 0$ for every simple closed curve C.Then for any two points $a, b \in R$. $\int_a^b f(z)dz$ is independent of path in the region.

Solution: Let $A = a$ and $B = b$ two points in the region and let ACD and ABD be two paths on the region from A to B. Then ABCDA is a simple closed curve in the region. By the given condition



$$\int_{ABCDA} f(z)dz = 0$$

Or, $\int_{ADB} + \int_{BCA} = 0$

$\Rightarrow$

$$\int_{ADB} = -\int_{BCA} = \int_{ACB}$$

Hence $\int_a^b f(z)dz$ is independent of the path in R.

Moreras Theorem (converse of Cauchys theorem): If $f(z)$ is continuous on a simple connected region R and if $\oint_C f(z)\,dz = 0$ for every simple closed curve C in R. Then $f(z)$ analytic in R.

Proof: Fix a point a in R.

For any $z \in R$, we define

$$F(z) = \int_a^z f(u)\,du$$

Choose $z + \Delta z$ in R. Then

$$F(z+\Delta z) = \int_a^{z+\Delta z} f(u)\,du$$

Now

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\left\{\int_a^{z+\Delta z} f(u)\,du - \int_a^z f(u)\,du\right\} - f(z)$$

$$= \frac{1}{\Delta z}\left\{\int_a^{z+\Delta z} f(u)\,du + \int_z^a f(u)\,du\right\} - f(z)$$

$$= \frac{1}{\Delta z}\left\{\int_z^a f(u)\,du + \int_a^{z+\Delta z} f(u)\,du +\right\} - f(z)$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} f(u)\,du - f(z)$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z}\big(f(u) - f(z)\big)du$$ [Here z is constant and u is variable]

Since $\oint_c f(u)du = 0$ for all simple closed curve C in R. So $\oint_z^{z+\Delta z}\{f(u) - f(z)\}$ is independent of path in R so long as the path inside R.

So we can choose a path as a straight line. We also choose $|\Delta z|$ sufficiently small so that the line z to $z + \Delta z$ lies in R. Since $f(u)$ is continuous at z, So for any u on this straight line and any $\varepsilon > 0$ There exist a $\delta > 0$ such that

$$|f(u) - f(z)| < \varepsilon \text{ whenever } |u - z| < \delta$$

Which will certainly hold if $|\Delta z| < \delta$. i.e., $|f(u) - f(z)| < \varepsilon$ whenever $|\Delta z| < \delta$

Hence

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \left|\frac{1}{\Delta z}\int_z^{z+\Delta z}\{f(u) - f(z)\}du\right| < \frac{\varepsilon}{|\Delta z|}|\Delta z|$$

$$= \varepsilon \qquad \text{whenever } |\Delta z| < \delta$$

$$\therefore \lim_{\Delta z \to 0}\frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$$

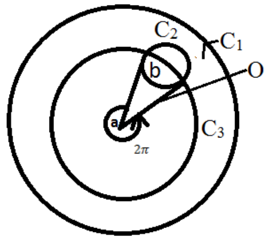
i.e.,

$$F'(z) = f(z)$$

$\therefore F(z)$ is differentiable at every point z in R. Hence $F(z)$ is analytic in R and so $F'(z) = f(z)$ is analytic in R.

Maximum modulas theorem: If $f(z)$ is analytic inside and on a simple closed curve C.Then the maximum value of |f(z)| ocuurs on C unless $f(z)$ is a constant.

Proof: Since $f(z)$ is analytic and hence continuous inside and on C, it follows that $|f(z)|$ has a maximum value M for at least one value of z inside or on C. Suppose this maximum value M not attained on the boundary of C but it is attained at an interior point a, is $|f(a)| = M$. Let C_1 be a circle inside C with centre at a enclosing the point b such that $|f(b)| < M$.



Suppose $|f(b)| = M - \varepsilon$ where $\varepsilon > 0$

Since $|f(z)|$ is continuous at b.Then for $\varepsilon > 0$, there exist $\delta > 0$ such that

$$\big||f(z)| - |f(b)|\big| < \frac{\varepsilon}{2} = M - \varepsilon + \frac{\varepsilon}{2} = M - \frac{\varepsilon}{2} \qquad \text{whenever } |z-b| < \delta$$

Construct a circle C_2 inside C with centre at b and radius δ. then for all z inside C_2

$$|f(z)| < M - \frac{\varepsilon}{2}$$

Now construct a circle C_3 with centre at a and radius $= |a-b|$. Then by gauss mean value theorem

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^{\alpha} f\left(a + re^{i\theta}\right) d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} f\left(a + re^{i\theta}\right) d\theta$$

i.e.,

$$|f(a)| \le \frac{1}{2\pi}\int_0^{\alpha}\left|f\left(a + re^{i\theta}\right)\right| d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi}\left|f\left(a + re^{i\theta}\right)\right| d\theta$$

i.e.,

$$|f(a)| \le \frac{1}{2\pi}\left(M - \frac{\varepsilon}{2}\right)\alpha + \frac{M}{2\pi}(2\pi - \alpha)$$

$$= M - \frac{\varepsilon\alpha}{4\pi}$$

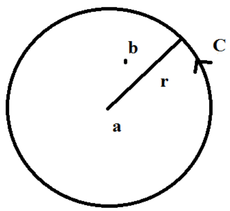
i.e.,

$$|f(a)| = M \le M - \frac{\varepsilon\alpha}{4\pi}$$

which is a contradiction. Hence the maximum value of $|f(z)|$ must occur on C.

Liouvilles Theorem: If $f(z)$ is analytic and bounded in the entire comple plan then $f(z)$ must be a constant.

Proof: Let a and b be any two points in the complex plane. Construct a circle C with centre at a and radius r enclosing b. Hence $f(z)$ is analytic inside and on C. So by Cauchys integral formula



$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{z-b} - \frac{1}{z-a}\right) f(z)\, dz$$

$$= \frac{1}{2\pi i}\oint \frac{b-a}{(z-a)(z-b)} f(z)\, dz$$

Choose r large enough so that $|a-b| < \frac{r}{2}$.

Then on C, $|z-a| = r$ and

$$|z-b| = |(z-a) - (b-a)| \ge |z-a| - |b-a| > r - \frac{r}{2} = \frac{r}{2}$$

Also $f(z)$ is bounded, so $|f(z)| < M$ for some +ve number M.

Therefore,

$$|f(b) - f(a)| = \left|\frac{1}{2\pi i}\oint_C \frac{(b-a)}{(z-a)(z-b)} f(z)\ \ dz\right| \le \frac{1}{2\pi}\frac{|b-a|M}{r.\frac{r}{2}}\ 2\pi r = \frac{2M|b-a|}{r}$$

i.e.,

$$|f(b) - f(a)| \le \frac{2M|b-a|}{r} \ \to 0 \ \ as \quad r \to \infty$$

Hence

$$f(b) = f(a)$$

i.e., $f(z)$ is constant.

Definition: A number α is called a zero of a function $f(z)$ if $f(\alpha) = 0$

Example: If $f(z) = z^3 + 5z + 6$ then $z = 2{,}3$ are zero of $f(z)$.

Definition: α is called a root of a polynomial equation $f(z) = 0$ if $f(\alpha) = 0$

Theorem: Every polynomial equation $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ here $n \ge 1$ has at least one root.

Proof: Suppose the equation has no root. Then $P(z) \ne 0$ for all complex number. Then $f(z) = \frac{1}{p(z)}$ is analytic on the enire complex plane. more over $f(z)$ is bounded for all z. In fact $|f(z)|$ tends to zero as $P(z) \to \propto$. Hence by Liouvilles theorem $f(z)$ is must be constant i.e., $\frac{1}{p(z)}$ is constant and so P(z) is constant. But this is a contradiction as $n \ge 1$

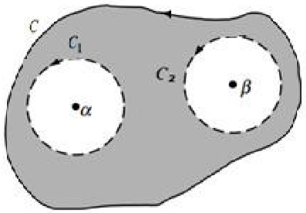
Therefore the equation has at least one root.

Argument Theorem: If $f(z)$ is analytic inside and on a simple closed curve C except for a pole at α of order p and only one zero at β of other n inside C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = n - p$$

Proof: Construct two circles C_1 and C_2 with centre at α and β respectively so that they lie insde C and they do not intersect each other.

Then $\frac{f'(z)}{f(z)}$ is analytic in the multiply connected region bounded by C_1, C_2 and C. So by cauchys Theorem for multiply connected region



$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz - \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f'(z)}{f(z)}\,dz = 0$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f'(z)}{f(z)}\,dz \quad \ldots\ldots\ldots\ldots(i)$$

Now for C_1, $f(z)$ has a pole at $z=\alpha$ of order p. Then

$$f(z) = \frac{F(z)}{(z-\alpha)^P} \quad \ldots\ldots\ldots\ldots(ii)$$

where $F(z)$ is analytic inside and on C_1 and contain no zero inside and on C_1.

Taking logarithms (ii) on both sides we get

$$\log f(z) = \log F(z) - p\ \log(z-\alpha)$$

Differentiating both sides we get

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{p}{z-\alpha}$$

Hence

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}dz - \frac{P}{2\pi i}\oint_{C_1} \frac{dz}{z-\alpha}$$

$$= 0 - \frac{P}{2\pi i}.2\pi i$$

$$= -P$$

We know

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}dz$$

Put $f(z) = 1$, then

$$1 = \frac{1}{2\pi i}\oint_C \frac{dz}{(z-a)}$$

For C_2, since $f(z)$ has a zero at β order n. so

$$f(z) = (z-P)^n G(z) \quad \ldots\ldots\ldots\ldots(iii)$$

where $G(z)$ is analytic, continuous, no zero inside and on C_2.

Taking logarithm on both side of (iii) we get

$$\log f(z) = \log G(z) + n\ \log(z-\beta)$$

Differentiating both sides we get

$$\frac{f'(z)}{f(z)} = \frac{G'(z)}{G(z)} + \frac{n}{z-\beta}$$

Hence

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_2} \frac{G'(z)}{G(z)}dz + \frac{n}{2\pi i}\oint_{C_2} \frac{dz}{z-\beta}$$

$$= 0 + \frac{n}{2\pi i}\ 2\pi i$$

$$= n$$

Therefore form (i)

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} = n - P$$

N.B.: Similarly we can show that if $f(z)$ is analytic inside on a simple closed curve C except a finite number of poles at $\alpha_1, \alpha_2, \ldots, \alpha_r$ inside C of order $p_1, p_2, \ldots p_r$ respectively and if $f(z)$ has zero at $\beta_1, \beta_2, \ldots, \beta_s$ inside C of order $n_1, n_2, \ldots n_s$ respectively then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{s} n_i - \sum_{i=1}^{r} P_i$$

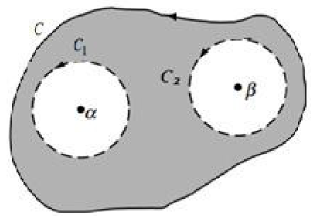
Corollary: If $f(z)$ is analytic inside and on a simple closed curve C except a pole at α inside C of order p and if $f(z)$ has zero at β inside C of order n then for any analytic function $g(z)$ inside and on C

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)}dz = n\,g(\beta) - p\,g(\alpha)$$

Proof: Constant two circles C_1 & C_2 inside C with centre at α & β repectively so that they do not intersect each other.

Then $g(z)\frac{f'(z)}{f(z)}$ is analytic in the multiple connected region bounded by C, C_1&C_2. Then by Cauchys theorem for multiple connected region we have

$$\oint_C g(z)\frac{f'(z)}{f(z)}dz = \oint_{C_1} g(z)\frac{f'(z)}{f(z)}dz + \oint_{C_2} g(z)\frac{f'(z)}{f(z)}dz \quad \ldots\ldots\ldots\ldots(i)$$



Since $f(z)$ has a pole at $z=\alpha$ of order p then

$f(z) = \frac{F(z)}{(z-\alpha)^P}$ where $F(z)$ is analytic inside and on C_1

Taking log on both sides we have

$$\log f(z) = \log f(z) - p\log(z-\alpha)$$

Differentiating we get

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - p/(z-\alpha)$$

$\Rightarrow$ $$g(z)\frac{f'(z)}{f(z)} = g(z)\frac{F'(z)}{F(z)} - \frac{g(z)\,p}{z-\alpha}$$

$\therefore$

$$\oint_{C1} g(z)\frac{f'(z)}{f(z)}dz = \oint_{C1} g(z)\frac{F'(z)}{F(z)}dz - \oint_{C2}\frac{g(z)p}{(z-\alpha)}dz$$

$$= 0 - p\oint_{C1}\frac{g(z)}{(z-\alpha)}dz = -p\,2\pi i\,g(\alpha)$$

$$= 2\pi i(-g(\alpha)P)$$

Similarly we can show that $\oint_{C2} g(z)\frac{f'(z)}{f(z)}dz = 2\pi i(-g(\beta)n)$

$\therefore$ From (i)

$$\oint_C g(z)\frac{f'(z)}{f(z)}dz = 2\pi i\{(g(\beta)n + p(-g(\alpha)p)\}$$

$\Rightarrow$

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)}dz = ng(\beta) - pg(\alpha)$$

[Proved]

Rouchis Theorem: If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C. Then $f(z)$ and $f(z)+g(z)$ haave same members of zeros inside C.

Proof: Let $F(z) = \frac{g(z)}{f(z)}$

So $g = Ff \Rightarrow g' = Ff' + F'f$

Here $|F(z)| = \left|\frac{g(z)}{f(z)}\right| < 1$ on C $\quad$ *as* $|g(z)| < |f(z)|$ *on* C

Now we get $|f(z)| \neq 0$ *on* C

Hence $F(z)$ is analytic on C

Let N_1 and N_2 be the number of zeros of f and $f+g$ respectively inside C

Then by argument theorem

$$N_2 - N_1 = \frac{1}{2\pi i}\oint_C \frac{f'+g'}{f+g}dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f' + Ff' + F'f}{f + fF}dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}dz$$

$$= \frac{1}{2\pi i}\oint_C \left\{\frac{f'(1+f) + F'}{f(1+F)} - \frac{f'}{f}\right\}dz$$

$$= \frac{1}{2\pi i}\oint_C \left\{\frac{f'}{f} + \frac{F'}{1+F} - \frac{f'}{f}\right\}dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}dz$$

Since $F(z)$ is analytic on C and $|F(z)| < 1$ so $1 + F \neq 0$ on C. Hence $\frac{F'}{1+F}$ is also analytic on C. Therefore by cauchys theorem

$$\oint_C \frac{F'}{1+F}dz = 0$$

Hence $N_2 - N_1 = 0$ $\quad$ i.e., $N_2 = N_1$

Therefore $f(z)$ and $f(z)+g(z)$ have some same number of zeros inside C.

Fundamental Theorem of Algebra: A polynomial equation $a_n z^n + a_{n-1}z^{n-1} + \cdots . + a_1 z + a_o = 0$ of degree $n \geq 1$ has exactly n roots

Proof: Let $f(z) = a_n z^n$ and $g(z) = a_{n-1}z^{n-1} + \cdots . + a_1 z + a_o = 0$

Consider a circle C with centre at origin and radius $r > 1$

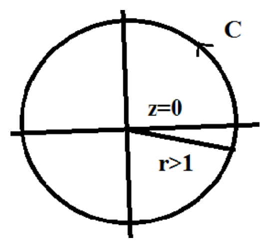
Here both $f(z)$ and $g(z)$ are analytic inside and on C

Now on C

$$\frac{|g(z)|}{|f(z)|} = \frac{|a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a|}{|a_n z^n|}$$

$$\leq \frac{|a_{n-1}||z^{n-1}| + \cdots + |a_1||z| + |a_o|}{|a_n||z^n|}$$

$$= \frac{|a_{n-1}||r^{n-1}| + \cdots + |a_1||r| + a_o}{|a_n||r^n|}$$



$$\leq \frac{|a_{n-1}|r^{n-1} + \cdots + |a_1|r^{n-1} + |a_o|r^{n-1}}{|a_n|r^n} \qquad | \because r > 1$$

$$= \frac{|a_{n-1}| + |a_{n-2}| + \cdots + |a_o|}{|a_n|r}$$

choosing r large enough we can make $\frac{|g(z)|}{|f(z)|} < 1$

Then by Rouchis theorem $f(z)$ and $f(z) + g(z)$ have some number of zeros inside C. But $f(z)$ has n mumber of zero inside C all located at origin. Therefore $f(z) + g(z)$ has n number of zero inside C.

i.e., $a_n z^n + a_{n-1}z^{n-1} + \cdots . + a_1 z + a_o = 0$ has exactly n number of roots.

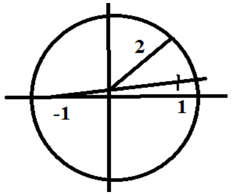
Example. Evaluate $\oint_C \frac{\cos \pi z}{z^2-1}\, dz$ where C is

(a) The circle $|z| = 2$
(b) The rectangle with vertices $-i, 2-i, 2+i, i$

Solution: We have $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$

So $\oint_C \frac{\cos \pi z}{z^2-1}\, dz = \frac{1}{2}\oint_C \frac{\cos \pi z}{z-1}\, dz - \frac{1}{2}\oint_C \frac{\cos \pi z}{z+1}\, dz$

(a) Let $f(z) = \cos \pi z$



Therefore

$$f(1) = \cos \pi = -1$$

And

$$f(-1) = \cos(-\pi) = -1$$

Hence by Cauchys integral formula

$$-1 = f(1) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-1}dz = \frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z-1}dz$$

$$\Rightarrow \frac{1}{2}\oint_C \frac{\cos\pi z}{z-1}dz = -\pi i$$

Similarly

$$(-1) = f(-1) = \frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z+1}dz$$

$$\Rightarrow \frac{1}{2}\oint_C \frac{\cos\pi}{z}dz = -\pi i$$

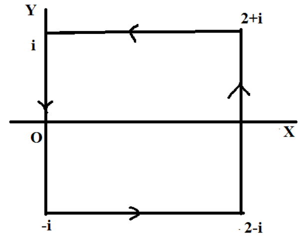
Hence

$$\oint_C \frac{\cos\pi z}{z^2-1}dz = -\pi i - (\pi i) = 0$$

(b) Since $z = -1$ lies outside C, So

$$\frac{1}{2}\oint_C \frac{\cos\pi z}{z+1}dz = 0$$

But $z = 1$ lies inside C
Let $f(z) = \cos\pi z$, Since $f(z)$ is analytic inside and on C



So by Cauchys integral formula

$$-1 = f(1) = \frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z-1}dz$$

$$\Rightarrow \frac{1}{2}\oint_C \frac{\cos\pi z}{z-1}dz = -\pi i$$

Hence $\oint_C \frac{\cos\pi z}{z^2-1}dz = -\pi i - 0 = -\pi i$

Example. Find the value of $\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz$ where C is the circle $|z| = 1$.

Solution : Let $f(z) = \sin^6 z$

Here $f(z)$ is analytic inside and on C. So by Cauchys integral formula

$$f''(\frac{\pi}{6}) = \frac{2!}{2\pi i}\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz$$

Now,

$$f(z) = \sin^6 z$$

$$f'(z) = 6\sin^5 z \cos z$$

$$f''(z) = 30\,sin^4 z cos^2 z - 6 sin^6 z$$

$$\therefore f''\left(\frac{\pi}{6}\right) = 30.\left(\frac{1}{2}\right)^4.\left(\frac{\sqrt{3}}{2}\right)^2 - 6\left(\frac{1}{2}\right)^6$$

$$= 30.\frac{1}{16}.\frac{3}{4} - \frac{6}{64}$$

$$= \frac{45}{32} - \frac{3}{32} = \frac{21}{16}$$

Hence

$$\frac{1}{\pi i}\oint_C \frac{sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{21}{16}$$

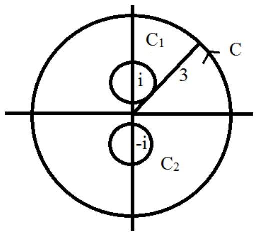
i.e., $\oint_C \frac{sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{21\pi i}{16}$

Example. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ where $t > 0$ and C is the circle $|z|$=3.

Solution: We have $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$

$$=\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z+i)^2(z-i)^2}dz$$

Here i & $-i$ lie inside C. Construct two circle C_1 & C_2 inside C with centre at i & $-i$ respectively so that they do not overlap each other. Then by Cauchys theorem



$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z+i)^2(z-i)^2}$$

$$=\frac{1}{2\pi i}\oint_{C1} \frac{e^{zt}\,dz}{(z+i)^2(z-i)^2} + \frac{1}{2\pi i}\oint_{C2} \frac{e^{zt}}{(z+i)^2(z-i)^2}dz$$

$$=I_1 + I_2 \text{ (Say)}$$

For I_1, let $f(z) = \frac{e^{zt}}{(z+i)^2}$

Here $f(z)$ is analytic inside and on C_1 and $z = i$ is inside C.

So by Cauchys integral formula,

$$f(i) = \frac{1}{2\pi i}\oint_{C1} \frac{e^{zt}}{(z+i)^2(z-i)^2}dz$$

Here

$$f(z) = \frac{(z+i)^2 + e^{zt} + e^{zt}2(z+i)}{(z+i)^4}$$

$$\therefore f(i) = \frac{-4te^{it} - e^{it}4i}{16} = \frac{-te^{it} - ie^{it}}{4}$$

For I_2, Let $g(z) = \frac{e^{zt}}{(z-i)^2}$

Here $g(z)$ is analytic inside and on C_2.

Also $z = -i$ lies inside C_2, So by Cauchys integral formula

$$g(-i) = \frac{1}{2\pi i}\oint_{C2} \frac{e^{zt}}{(z+i)^2(z-i)^2}dz$$

Here

$$g(z) = \frac{(z-i)^2 + e^{zt} - e^{zt}(z-i)}{(z+i)^4}$$

Hence $g(-i) = \frac{-4t\,e^{it}+4ie^{it}}{16} = \frac{-te^{-it}+ie^{-it}}{4}$

Therefore

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2} = I_1 + I_2$$

$$= f(i) + g(-i)$$

$$= \frac{-t\left(e^{it}+e^{-it}\right) - i\left(e^{it}-e^{-it}\right)}{4}$$

$$= \frac{-2tcost + 2sint}{4}$$

$$= \frac{1}{2}(sint - t\,cost)$$

Example. Let $f(z) = \frac{(z^2+1)^2(z-3)(z+5)^3}{(z^2+2z+2)^3(z-6)}$

Evaluate

(i) $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$ and (ii) $\frac{1}{2\pi i}\oint_C \frac{(z^2+z)f'(z)}{f(z)}dz$

Where C is the circle $|z| = 4$.

Solution:

(1) put $z^2+1=0$ the $z=\pm i$

$z-3=0$ then $z=3$

$z+5=0$ then $z=-5$

So $f(z)$ has zero at $z=i$ of order 2

at $z=-i$ of order 2

at $z=3$ of order 1

at $z=-5$ of order 3

But only i,-i,3 lie inside C

Hence the number of zero inside $C=2+2+1=5$

Again put $z^2+2z+2=0$

Then $z=-1\pm i$

Put $z-6=0$ then $z=6$

So $f(z)$ has pole at $z=-1+i$ of order 3

at $z=-1-i$ order 3

at $z=6$ of order 1

But $-1+i$ and $-1-i$ lie inside C

Therefore the member of poles inside $C=3+3=6$

Hence by argument theorem

$\frac{1}{2\pi i}\oint_C \frac{f\prime(z)}{f(z)}dz=5-6=-1$

(Ans)

(ii) Put $z^2+1=0$ then z=$\pm i$

$z-3=0$ then $z=3$

$z+5=0$ then $z=-5$

So $f(z)$ has zero at $z=i$ of order 2

at $z=-i$ of order 2

at $z=3$ of order 1

at $z=-5$ of order 3

But only $i,-i,3$ lie inside C

Again put $z^2+2z+2=0 \Rightarrow z=-1\pm i$

$z-6=0 \;\Rightarrow z=6$

So $f(z)$ has zero at $z=-1+i$ of order 3

at $z=-1-i$ of order 3

at $z=6$ of order 1

But only $-1+i$ and $-1-i$ lie inside C

Now put $g(z)=z^2+z$ Here $g(z)$ is analytic inside and on C. therefore by argument theorem

$$\frac{1}{2\pi i}\oint_C (z^2+2)\frac{f'(z)}{f(z)}dz=2g(i)+2g(-i)+1.g(2)-3g(-1+i)-3g(-1-i)$$

$$=2(-1+i)+2(-1-i)+12-3[-i-1)-3(i-1)$$

$$=-4+12+3+3$$

=14

Ex. Prove that all roots of $z^7-5z^3+12=0$ lie between the circle $|z|=1$ and $|z|=2$.

Proof: Let $f(z)=z^7$ and $g(z)=-5z^3+12$

Let C_1 represent the circle $|z|=1$

And C_2 represent the circle $|z|=2$

Now on C_2 $|g(z)|=|-5-z^3+12|\le 5|z|^3+12=5.2^3+12$

$$=52<2^7=|f(z)|$$

Since both $f(z)$ and $g(z)$ are analytic inside and on C_2. So by Rouchis theorem $f(z)$ and $f(z)+g(z)$ have some number of inside C_2.Since $f(z)$ has & zeroes inside C_2 (all located origin)

So $f(z)+g(z)=z7-5z3+12$ has 7 zeroes inside C_2

For C_1 let $f(z)=12$ and $g(z)=z^7-5z^3$

Now $|g(z)| \le |z|^7 + 5|z|^3 = 1 + 5 = 6 < 12 = |f(z)|$

Also both $f(z)$ and $g(z)$ are analytic inside and on C_1 So by Rouchis theorem

$f(z) + g(z)$ have some number of zeroes inside C_1.But $f(z)$ has no zero inside C_1. Therefore $f(z) + g(z) = z^7 = 5z^3 + 12$ has no zero onside C_1. i.e., all zero of $z^7 - 5z^3 + 12$ is outside of C_1

∴All zeroes of $z^7 - 5z^3 + 12$ lie between the circle $|z| = 1$ & $|z| = 2$.

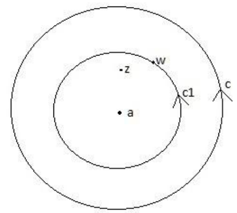
Taylor's Theorem

Taylor Theorem: If $f(z)$ is analytic inside a circle C with centre at a then for any z inside C

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Proof: Construct a circle C, inside C with Centre at a inclosing z. Then $f(z)$ is analytic inside and on C. Then by Cauchy's integral formula,

$$f(\mathrm{z}) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw$$



Now

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\{1-\left(\frac{z-a}{w-a}\right)\}} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

Since on C, $\left|\frac{z-a}{w-a}\right| < 1$

So

$$\frac{1}{w-z} = \frac{1}{w-a}\left\{1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\cdots\cdots\cdots\right\} \quad \dots\dots\dots\dots\dots (1)$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\cdots + \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n\left\{1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\cdots\right\}\right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots\cdots\cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \cdots\cdots\cdots + \left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z} \quad \text{[By equation (1)]}$$

So

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-a} + \frac{(z-a)f(W)}{(w-a)^2} + \frac{(z-a)^2 f(w)}{(w-a)^3} + \cdots\cdots\cdots + \frac{(z-a)^{n-1}}{(w-a)^n}f(w) + \left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}$$

Or

$$,\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}dw$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}dw + \frac{(z-a)}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}dw + \frac{(z-a)^2}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^3}dw + \cdots\cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}dw + u_n$$

Where $u_n = \frac{1}{2\pi i}\oint_{C_1}\frac{(z-a)^n}{(w-a)^n}\frac{f(w)}{w-z}dw$

Now using Cauchy's integral formula we get,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots\cdots + \frac{(z-a)^{n-1}}{(n-1)!}f^{n-1}(a) + u_n$$

$$\left(\text{Since, } f^n(a) = \frac{n!}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw\right)$$

We need to show that,

$\lim_{n\to\infty} u_n = 0$

On c_1 let $\left|\frac{z-a}{w-a}\right| = \beta < 1$

$|w-z| = |(w-a)-(z-a)| > |w-a| - |z-a| = r_1 - |z-a|$

Where r_1 is the radius of c_1

$\therefore \left|\frac{1}{w-z}\right| < \frac{1}{r_1 - |z-a|}$

Since $f(w)$ is analytic inside and on c_1. So $|f(w)| \le M$ for some positive number M.

Hence $|u_n| = \left|\frac{1}{2\pi i}\oint_{c_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}dw\right| < \frac{1}{2\pi}\frac{\beta^n M}{r_1-|z-a|}2\pi r_1$

$\to 0$ as n→ ∞ | since β<1

Therefore $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots\cdots\cdots$

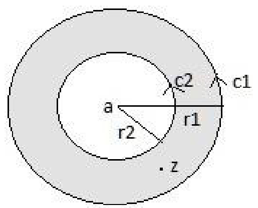
Laurent's Theorem: If f (z) is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circle c_1 and c_2 with Centre at a and radius r_1 and r_2 respectively. Then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_{n(z-a)^n} + \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n}$$

Where $a_n = \frac{1}{2\pi i}\oint_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw$

And $a_{-n} = \frac{1}{2\pi i}\oint_{c_2}\frac{f(w)}{(w-a)^{-n+1}}dw$

Proof: Since z is inside R, so by Cauchy's integral formula for multiple connected region



$$f(z) = \frac{1}{2\pi i}\oint_{c_1}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\oint_{c_2}\frac{f(w)}{w-z}dw - - - - - - - - (1)$$

For the first integral w is on c_1.

Then $\left|\frac{z-a}{w-a}\right| < 1$

So $\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$

$\therefore \frac{1}{w-z} = \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\cdots\cdots\right]\cdots\cdots\cdots(1)$

$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \cdots\cdots + \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)\left\{1 + \frac{z-a}{w-a} + \cdots\cdots\right\}\right]$

$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots\cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z}$ [by (1)]

Hence $\frac{1}{2\pi i}\oint_{c_1}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{c_1}\frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\oint_{c_1}\frac{f(w)}{(w-a)^2}dw + \cdots\cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{c_1}\frac{f(w)}{(w-a)^n}dw + u_n$

Where $u_n = \frac{1}{2\pi i}\oint_{c_1}\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}dw$

Thus $\frac{1}{2\pi i}\oint_{c_1}\frac{f(w)}{w-z}dw = a_0 + a_1(z-a) + \cdots\cdots + a_{n-1}(z-a)^{n-1} + u_n$

Where $a_n = \frac{1}{2\pi i}\oint_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw$

For the second integral when w is on c_2.

Then $\left|\frac{w-a}{z-a}\right| < 1$

Hence $-\frac{1}{w-z} = \frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = \frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$

$= \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots\cdots\cdots\right]$

$= \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots\cdots + \left(\frac{w-a}{z-a}\right)^{n-1} + \left(\frac{w-a}{z-a}\right)^n\left\{1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots\cdots\right\}\right]$

Or, $-\frac{1}{w-z} = \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \cdots\cdots + \frac{(w-a)^{n-1}}{(z-a)^n} + \left(\frac{w-a}{z-a}\right)^n\frac{1}{z-w}$

So, $-\frac{1}{2\pi i}\oint_{c_2}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i(z-a)}\oint_{c} f(w)dw + \frac{1}{2\pi i(z-a)^2}\oint_{c_2} f(w)(w-a)dw + \cdots$

$\cdots + \frac{1}{2\pi i(z-a)^n}\oint_{c_2}(w-a)^{n-1}f(w)dw + v_n$

Where, $v_n = \frac{1}{2\pi i}\oint_{c_2}\left(\frac{w-a}{z-a}\right)^n\frac{f(w)}{z-a}dw$

Thus $-\frac{1}{2\pi i}\oint_{c_2}\frac{f(w)}{w-z}dw = \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\cdots + \frac{a_{-n}}{(z-a)^n} + v_n$

Where $a_{-n} = \frac{1}{2\pi i}\oint_{c_2}\frac{f(w)}{(w-a)^{-n+1}}dw$ $n = 1,2,3, \ldots\ldots\ldots$

Hence by (1),

$$f(z) = a_0 + a_1(z-a) + \cdots\cdots + a_{n-1}(z-a)^{n-1} + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\cdots + \frac{a_{-n}}{(z-a)^n} + u_n + v_n \cdots (2)$$

Where w is on c_1, let $\left|\frac{z-a}{w-a}\right| = \beta < 1$

Since $f(z)$ is analytic inside and on the region bounded by c_1 and c_2 so $|f(z)|$ <M for some positive number.

Also on c_1 $|w-z| = |(w-a)-(z-a)|$

$\ge |w-a| - |z-a|$

$$= r_1 - |z-a|$$

So $\left|\frac{1}{w-z}\right| \leq \frac{1}{r_1-|z-a|}$

Therefore $|\frac{1}{2\pi i}\oint_{C_1}\frac{(z-a)^n}{(w-a)^n}\frac{f(w)}{w-z}dw| \leq \frac{1}{2\pi}\frac{\beta^n M}{r_1-|z-a|}2\pi r_1 \to 0$ $\quad as \quad n\to\infty \quad |since\ \beta<1$

$$\lim_{n\to\infty} u_n = 0$$

Similarly we kan show that $\lim_{n\to\infty} V_n = 0$

Hence from (2)

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\cdots\cdots$$

That Is, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^2}$

Note: We can also write the Laurent's expression as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

Where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)^{n+1}}dw$ and C is any simple closed curve, the region enclosing the point $z=a$.

- In Laurent's expansion $\sum_{n=0}^{\infty} a_n(z-a)^n$ is called analytic part and $\sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n}$ is called principal part.
- If $f(z)$ has no singularity inside and on C_1(i.e., $f(z)$ is analytic) then the series doesn't possess any principal part. Thus the Laurent's explanation concide with Taylor's explanation, the principal part occurs only because of singularity of function.
- In Laurent's expansion of a function $f(z)$, if number of terms in the principal part is finite say n, then the singularity is pole of order n.
- If the number of terms is infinite then the singularity is an essential singularity.
- If the principal part vanishes then the singularity is called a removable singularity.

Example. Find Laurent's series about the indicated singularity. For each function name the singularity in each case and give the region of convergence of each series.

(a) $f(z) = \frac{e^{3z}}{(z-2)^4};\quad z=2$

(b) $f(z) = (z-4)sin\frac{1}{z+3};\quad z=-3$

(c) $f(z) = \frac{z-sin\,z}{z^2};\quad z=0$

Solution: (a) put $z-2=u$, then $z=u+2$

$$\therefore f(z) = \frac{e^{3z}}{(z-2)^4} = \frac{e^{3(u+2)}}{u^4} = \frac{e^6}{u^4}e^{3u}$$

$$= \frac{e^6}{u^4}[1 + 3u + \frac{(3u)^2}{2!} + \frac{(3u)^3}{3!} + \cdots\cdots]$$

$$= e^6[\frac{1}{u^4} + \frac{3}{u^3} + \frac{9}{2}\frac{1}{u^2} + \frac{9}{2}\frac{1}{u} + \frac{27}{8} + \frac{81}{40}u + \cdots\cdots]$$

$$= e^6[\frac{1}{(z-2)^4} + \frac{3}{(z-2)^3} + \frac{9}{2}\frac{1}{(z-2)^2} + \frac{9}{2}\frac{1}{z-2} + \frac{27}{8} + \frac{81}{40}(z-2) + \cdots\cdots\cdots]$$

This is the Laurent's series of the function around $z=2$.

Here the principal part consists of only 4 terms. So $z=2$ is a pole of order 4.

The series is convergent only if

$1 + 3u + \frac{(3u)^2}{2!} + \cdots\cdots\cdots$ is convergent, provided $u \neq 0$

Here $a_n = \frac{(3u)^{n-1}}{(n-1)!}$

$$a_{n+1} = \frac{(3u)^n}{n!}$$

$$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3u)^n}{n!}\times\frac{(n-1)!}{(3u)^{n-1}}\right| = \frac{3}{n}|u|$$

Now $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{3}{n}|u| = 0 < 1$

Hence by ratio test the series is convergent for all $u \neq 0$

That is the series is convergent for all $z \neq 2$

(b) Here $f(z) = (z-4)sin\frac{1}{z+3}\quad ;\quad z=-3$

Put $z+3=u$ $\quad$ Then $\quad z=u-3$

$$\therefore f(z) = (u-7)sin\frac{1}{u}$$

$$= (u-7)\{\frac{1}{u} - \frac{1}{3!\,u^3} + \frac{1}{5!\,u^5} - \cdots\cdots\cdots\}$$

$$= 1 - \frac{7}{u} - \frac{1}{3!\,u^2} + \frac{7}{3!\,u^3} + \frac{1}{5!\,u^4} - \frac{7}{5!\,u^5} - \cdots\cdots\cdots$$

$$= 1 - \frac{7}{z+3} - \frac{1}{3!\,(z+3)^2} + \frac{7}{3!\,(z+3)^3} + \frac{1}{5!\,(z+3)^4} - \frac{7}{5!\,(z+3)^5}\cdots\cdots\cdots$$

Which is the required Laurent's series around $z=-3$

Since the number of terms of the principal part in the expansion is infinite, so $z=-3$ is an essential singularity.

This series is convergent only if

$\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}-\cdots\cdots\cdots$ is convergent

$$a_n=\frac{(-1)^{n-1}}{(2n-1)!\,u^{2n-1}}$$

$$a_{n+1}=\frac{(-1)^{n}}{(2n+1)!\,u^{2n+1}}$$

$$\therefore\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(-1)^n}{(2n+1)!\,u^{2n+1}}\times\frac{(2n-1)!\,u^{(2n-1)}}{(-1)^{n-1}}\right|=\frac{1}{2n(2n+1)|u|^2}$$

Now $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{1}{2n(2n+1)|u|^2}=0<1$ provided $u\neq 0$

Hence by ratio test the series is convergent for all $u\neq 0$.

Is the Laurent series is convergent for all $z\neq -3$.

(c)Given that $f(z)=\frac{z-sinz}{z^2}$; $z=0$

We get $\frac{z-sinz}{z^2}=\frac{1}{z^2}\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\cdots\cdots\right)\}$

$=\frac{1}{z^2}(\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-\cdots\cdots\cdots)$

$=\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-\cdots\cdots\cdots$

Which is the Laurent series for the function around $z=0$.

Since the series has no principal part so $z=0$ is a removable singularity.

Here $a_n=(-1)^{n-1}\frac{z^{2n-1}}{(2n+1)!}$

$$a_{n+1}=(-1)^n\frac{z^{2n+1}}{(2n+3)!}$$

$$\therefore\left|\frac{a_{n+1}}{a_n}\right|=\frac{|z|^2}{(2n+3)(2n+2)}$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0<1$

So by ratio test the series converges for all z.

Example. Expand $f(z)=\frac{z}{z^2+5z+6}$ in a Laurent series valid for

(a) $2<|z|<3$
(b) $|z|>3$
(c) $1<|z+1|<2$
(d) $|z|<2$

Solution: Here $f(z)=\frac{z}{z^2+5z+6}=\frac{z}{(z+2)(z+3)}=\frac{2}{z+3}-\frac{2}{z+2}$

(a) Given condition is $2<|z|<3$

When $2<|z|$

$$\frac{2}{z+2}=\frac{2}{z(1+\frac{2}{z})}=\frac{2}{z}(1+\frac{2}{z})^{-1}$$

$$=\frac{2}{z}(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\cdots\cdots)$$

$$=\frac{2}{z}-\frac{4}{z^2}+\frac{8}{z^3}-\frac{16}{z^4}+\cdots\cdots\cdots$$

When $|z|<3$

$$\frac{3}{z+3}=\frac{3}{3(1+\frac{z}{3})}=(1+\frac{z}{3})^{-1}$$

$$=1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\frac{z^4}{81}-\cdots\cdots\cdots$$

Hence the Laurent series is

$$\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\cdots\cdots\right)-\left(\frac{2}{z}-\frac{4}{z^2}+\frac{8}{z^3}-\frac{16}{z^4}+\cdots\cdots\cdots\right)$$

$=\cdots\cdots\cdots+\frac{16}{z^4}-\frac{8}{z^3}+\frac{4}{z^2}-\frac{2}{z}+1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\cdots\cdots$

(b) Given region is $|z|>3$

Now $\frac{2}{z+2}=\frac{2}{z(1+\frac{2}{z})}=\frac{2}{z}(1+\frac{2}{z})^{-1}$

$$=\frac{2}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\cdots\cdots\right)$$

$$=\frac{2}{z}-\frac{4}{z^2}+\frac{8}{z^3}-\frac{16}{z^4}+\cdots\cdots\cdots$$

$$\frac{3}{z+3}=\frac{3}{z(1+\frac{3}{z})}=\frac{3}{z}(1+\frac{3}{z})^{-1}$$

$$=\frac{3}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}\cdots\cdots\right)$$

$=\frac{3}{z}-\frac{9}{z^2}+\frac{27}{z^3}-\frac{81}{z^4}+\cdots\cdots\cdots$

So the Laurent series is

$$\frac{3}{z}-\frac{9}{z^2}+\frac{27}{z^3}-\frac{81}{z^4}+\cdots\cdots\cdots-\left(\frac{2}{z}-\frac{4}{z^2}+\frac{8}{z^3}-\frac{16}{z^4}+\cdots\cdots\cdots\right)$$

$$=\frac{1}{z}-\frac{5}{z^2}+\frac{19}{z^3}-\frac{65}{z^4}+\cdots\cdots\cdots$$

(c) Given region is $1<|z+1|<2$

Put $z+1=u$ Then $1<|u|<2$

$$\therefore f(z)=\frac{3}{z+3}-\frac{2}{z+2}=\frac{3}{u+2}-\frac{2}{u+1}$$

When $1<|u|$

Then $\frac{2}{u+1}=\frac{u}{2}\left(1+\frac{1}{u}\right)^{-1}$

$=\frac{2}{u}\left(1-\frac{1}{u}+\frac{1}{u^2}-\frac{1}{u^3}+\cdots\cdots\cdots\right)$

$=\frac{2}{u}-\frac{2}{u^2}+\frac{2}{u^3}-\frac{2}{u^4}+\cdots\cdots\cdots$

When $|u|<2$; Then

$$\frac{3}{u+2}=\frac{3}{2}\left(1+\frac{u}{2}\right)^{-1}$$

$=\frac{3}{2}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8}+\cdots\cdots\cdots\right)$

$=\frac{3}{2}-\frac{3u}{u}+\frac{3u^2}{8}-\frac{3u^3}{16}+\cdots\cdots\cdots$

Hence the series is

$$\left(\frac{3}{2}-\frac{3u}{u}+\frac{3u^2}{8}-\frac{3u^3}{16}+\cdots\cdots\cdots\right)-\left(\frac{2}{u}-\frac{2}{u^2}+\frac{2}{u^3}-\frac{2}{u^4}+\cdots\cdots\cdots\right)$$

$$=\cdots\cdots\cdots+\frac{2}{u^4}-\frac{2}{u^3}+\frac{2}{u^2}-\frac{2}{u}+\frac{3}{2}-\frac{3u}{4}+\frac{3u^2}{8}-\frac{3u^3}{16}+\cdots\cdots\cdots$$

$$=\cdots\cdots\cdots+\frac{2}{(z+1)^4}-\frac{2}{(z+1)^3}+\frac{2}{(z+1)^2}-\frac{2}{(z+1)}+\frac{3}{2}-\frac{3(z+1)}{4}+\frac{3(z+1)^2}{8}-\frac{3(z+1)^3}{16}+\cdots\cdots\cdots$$

(d) Given region is $|z|<2$

Now $\frac{3}{z+3}=\frac{3}{3\left(1+\frac{z}{3}\right)}=1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\cdots\cdots$

$$\frac{2}{z+2}=\frac{2}{2\left(1+\frac{z}{2}\right)}=1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\cdots\cdots$$

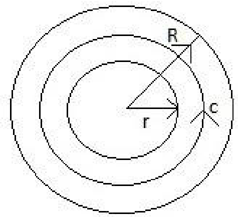
Hence the series is

$$1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\cdots\cdots-\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\cdots\cdots\right)$$

$$=\frac{z}{6}-\frac{5}{36}z^2+\frac{19}{196}z^3-\cdots\cdots\cdots\cdots$$

Example. Show that $\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_{n=1}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right)$

Where $a_n=\frac{1}{2\pi}\int_0^{2\pi}\cosh(2cos\theta)\,cosn\theta\;d\theta$



Solution: Here $\cosh\left(z+\frac{1}{z}\right)$ is

analytic for every finite value of z except at $z=0$

So it is analytic in the ring shaped region $r<|z|<R$ where r is sufficiently

small and R is sufficiently large. So by Laurent's theorem $\cosh\left(z+\frac{1}{z}\right)=\sum_{n=0}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\frac{a_{-n}}{z^n}$

Where $a_n=\frac{1}{2\pi i}\oint_c\frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}dz,\qquad a_{-n}=\frac{1}{2\pi i}\oint_c\frac{\cosh\left(z+\frac{1}{z}\right)}{z^{-n+1}}dz$

Where c is a simple closed curve in the annulus (ring shaped region) enclosing the region.

Now choose c as the unit circle with centre at origin.

Then on c $\qquad z=e^{i\theta}$

$dz=ie^{i\theta}d\theta$

$z+\frac{1}{z}=e^{i\theta}+e^{-i\theta}=2\cos\theta$

So $a_n=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\cos\text{h}\,(2cos\theta)}{e^{i(n+1)\theta}}ie^{i\theta}d\theta$

$$=\frac{1}{2\pi}\int_0^{2\pi}\cos\text{h}\,(2cos\theta)e^{-in\theta}\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\cos\text{h}(2cos\theta)\,cosn\theta\;d\theta-\frac{i}{2\pi}\int_0^{2\pi}\cos\text{h}(2cos\theta)\,sin\,n\theta\;d\theta$$

Now $\int_0^{2\pi}\cos\text{h}(2cos\theta)\,sin\,n\theta\;d\theta$

$=\int_0^{2\pi}\cos\text{h}(2\cos(2\pi-\theta))\,sin\,n(2\pi-\theta)\;d\theta$

$=\int_0^{2\pi}\cos\text{h}\,(2cos\theta)(-\sin n\theta)\;d\theta$

$=-\int_0^{2\pi}\cos h(2\cos\theta)\sin n\theta\;d\theta$

$$\therefore \int_0^{2\pi} \cos h(2\cos\theta)\sin n\theta \, d\theta = 0$$

So $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos h(2\cos\theta)\cos n\theta \, d\theta$

Also $a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cos h(2\cos\theta)\cos(-n\theta)\, d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} \cos h(2\cos\theta)\cos n\theta \, d\theta$

$= a_n$ for $n = 1,2,3,\cdots\cdots\cdots$

Hence $\cos h\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{a_n}{z^n}$

$= a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{1}{z^n}\right)$

(Proved)

12

Fourier Series

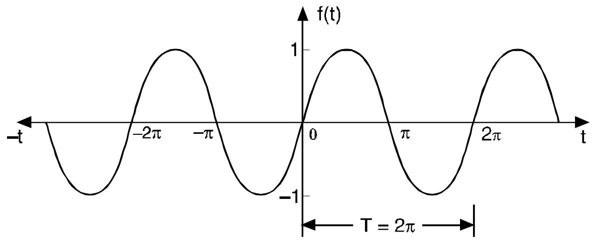
12.1 PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t+T) = f(t+2T) = \ldots.$ then T is called the period of the function $f(t)$.

For example :

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ so $\sin x$ is a periodic function with the period 2π. This is also called sinusoidal periodic function.



12.2 FOURIER SERIES

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form.

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

is called the *Fourier series, where* $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3 \ldots b_n \ldots$ are constants.

A periodic function $f(x)$ can be expanded in a Fourier Series. The series consists of the following:

(*i*) A constant term a_0 (called d.c. component in electrical work).

(*ii*) A component at the fundamental frequency determined by the values of a_1, b_1.

(*iii*) Components of the harmonics (multiples of the fundamental frequency) determined by $a_2, a_3 \ldots b_2, b_3 \ldots$. And $a_0, a_1, a_2 \ldots, b_1, b_2 \ldots$ are known as *Fourier coefficients* or Fourier constants.

12.3. DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

If the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single-valued (2) is bounded

(3) has at most a finite number of maxima and minima.

(4) has only a finite number of discontinuous

(5) is $f(x+2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$, then

$$S_p(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1}^{P} b_n \sin nx$$

converges to $f(x)$ as $P \to \infty$ at values of x for which $f(x)$ is continuous and to $\frac{1}{2}[f(x+0)+f(x-0)]$ at points of discontinuity.

12.4. ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor's series).

2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.

4. Fourier series of a discontinuous function is not uniformly convergent at all points.

5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

12.5 USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

(*i*) $\int_0^{2\pi} \sin nx\, dx = 0$ (*ii*) $\int_0^{2\pi} \cos nx\, dx = 0$

(*iii*) $\int_0^{2\pi} \sin^2 nx\, dx = \pi$ (*iv*) $\int_0^{2\pi} \cos^2 nx\, dx = \pi$

(*v*) $\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = 0$ (*vi*) $\int_0^{2\pi} \cdot \cos nx \cos mx\, dx = 0$

(*vii*) $\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = 0$ (*viii*) $\int_0^{2\pi} \sin nx \cdot \cos nx\, dx = 0$

(*ix*) $[uv] = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

where $v_1 = \int v\, dx$, $v_2 = \int v_1\, dx$ and so on. $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on

(*x*) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \in I$

12.6 DETERMINATION OF FOURIER COEFFICIENTS (EULER'S FORMULAE)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots \quad \ldots(1)$$

(*i*) **To find a_0:** Integrate both sides of (1) from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\, dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x\, dx + a_2 \int_0^{2\pi} \cos 2x\, dx + \ldots + a_n \int_0^{2\pi} \cos nx\, dx + \ldots$$
$$+ b_1 \int_0^{2\pi} \sin x\, dx + b_2 \int_0^{2\pi} \sin 2x\, dx + \ldots + b_n \int_0^{2\pi} \sin nx\, dx + \ldots$$
$$= \frac{a_0}{2}\int_0^{2\pi} dx, \quad \text{(other integrals } = 0 \text{ by formulae } (i) \text{ and } (ii) \text{ of Art. 12.5)}$$

$$\int_0^{2\pi} f(x)\, dx = \frac{a_0}{2} 2\pi, \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx \quad \ldots(2)$$

(*ii*) **To find a_n:** Multiply each side of (1) by $\cos nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\cos nx\, dx = \frac{a_0}{2}\int_0^{2\pi} \cos nx\, dx + a_1 \int_0^{2\pi} \cos x \cos nx\, dx + \ldots + a_n \int_0^{2\pi} \cos^2 nx\, dx \ldots$$
$$+ b_1 \int_0^{2\pi} \sin x \cos nx\, dx + b_2 \int_0^{2\pi} \sin 2x \cos nx\, dx + \ldots$$
$$= a_n \int_0^{2\pi} \cos^2 nx\, dx = a_n \pi \quad \text{(Other integrals = 0, by formulae on Page 851)}$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \quad \ldots(3)$$

By taking $n = 1, 2 \ldots$ we can find the values of $a_1, a_2 \ldots$.

(*iii*) **To find b_n:** Multiply each side of (1) by $\sin nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\sin nx\, dx = \frac{a_0}{2}\int_0^{2\pi} \sin nx\, dx + a_1 \int_0^{2\pi} \cos x \sin nx\, dx + \ldots + a_n \int_0^{2\pi} \cos nx \sin nx\, dx + \ldots$$
$$+ b_1 \int_0^{2\pi} \sin x \sin nx\, dx + \ldots + b_n \int_0^{2\pi} \sin^2 nx\, dx + \ldots$$
$$= b_n \int_0^{2\pi} \sin^2 nx\, dx$$

(All other integrals = 0, Article No. 12.5)

$$= b_n \pi$$

$$\therefore \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \quad \ldots(4)$$

Note: To get similar formula of a_0, $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1. *Find the Fourier series representing*

$$f(x) = x, \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Solution. Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \ldots \quad \ldots(1)$

Hence $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

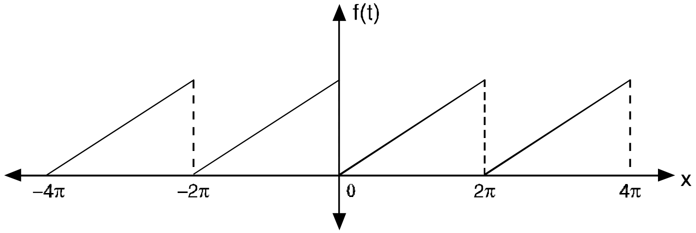
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of $a_0, a_1, a_2 ..., b_1, b_2 ...$ in (1)

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right] \quad \textbf{Ans.}$$



Example 2. *Given that* $f(x) = x + x^2$ *for* $-\pi < x < \pi$, *find the Fourier expression of* $f(x)$.

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ (*U.P. II Semester, Summer 2003*)

Solution. Let $$x + x^2 = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots \quad \ldots(1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (2x+1)\frac{(-\cos nx)}{n^2} + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(\frac{-\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3}\cdot + (-\pi+\pi^2)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right]$$
$$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \ldots\right] \quad \ldots(2)$$

Put $$x = \pi \text{ in (2)}, \ \pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right] \quad \ldots(3)$$

Put $$x = -\pi \text{ in (2)}, \ -\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right] \quad \ldots(4)$$

Adding (3) and (4) $$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \textbf{Ans.}$$

Exercise 12.1

1. Find a Fourier series to represent, $f(x) = \pi - x$ for $0 < x < 2\pi$.

Ans. $2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots + \frac{1}{n}\sin nx + \ldots\right]$

2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(*Mysore 1997, Osmania 1995*)

Ans. $-\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right]$

3. Find a Fourier series to represent: $f(x) = x\sin x$, for $0 < x < 2\pi$.

(*A.M.I.E.T.E., Summer 1997, Madras 1997, Mysore 1995*)

Ans. $-1 + \pi\sin x - \frac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} + \ldots\right]$

4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh\pi}$.

Ans. $\frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2} - \frac{1}{1^2+1}\cos x + \frac{1}{2^2+1}\cos 2x \quad - \frac{1}{3^2+1}\cos 3x + \ldots\right)\right.$
$\left. + \frac{1}{1^2+1}\sin x - \frac{2}{2^2+1}\sin 2x + \frac{3}{3^2+1}\sin 3x \ldots\right] \quad \frac{\pi}{\sinh\pi} = 1 + 2\left[-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \ldots\right]$

5. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 \le x < 2\pi$. (*Nagpur 1997*)

Ans. $\dfrac{1-e^{-2\pi}}{\pi}\left[\dfrac{1}{2}+\dfrac{1}{2}\cos x+\dfrac{1}{5}\cos 2x+\dfrac{1}{10}\cos 3x+\dfrac{1}{2}\sin x+\dfrac{2}{5}\sin 2x+\dfrac{3}{10}\sin 3x+\ldots\right]$

6. If $f(x)=\left(\dfrac{\pi-x}{2}\right)^2$, $0<x<2\pi$, show that $f(x) = \dfrac{\pi^2}{12}+\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$ (*Madras 1998*)

7. Prove that $x^2 = \dfrac{\pi^2}{3}+4\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{\cos nx}{n^2}$, $-\pi<x<\pi$.

Hence show that (*i*) $\Sigma\dfrac{1}{n^2}=\dfrac{\pi^2}{6}$ (*Madras 1997, Mangalore 1997, Warangal 1996*)

(*ii*) $\Sigma\dfrac{1}{(2n-1)^2}=\dfrac{\pi^2}{8}$ (*Mangalore 1997*) (*iii*) $\Sigma\dfrac{1}{n^4}=\dfrac{\pi^4}{90}$ (*Madras 1997*)

8. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ by $f(x) = \dfrac{(3x^2-6x\pi+2\pi^2)}{12}$.

Prove that $f(x) = \sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$ and hence show that $\dfrac{\pi^2}{6} = 1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$

12.7 FUNCTION DEFINED IN TWO OR MORE SUB-RANGES

Example 3. *Find the Fourier series of the function*

$$f(x) = \begin{cases} -1 & for \quad -\pi < x < -\dfrac{\pi}{2} \\ 0 & for \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ +1 & for \quad \dfrac{\pi}{2} < x < \pi. \end{cases}$$

Solution. Let $f(x) = \dfrac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots+b_1\sin x+b_2\sin 2x+\ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}0\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}1\,dx$$

$$= \frac{1}{\pi}\Big[-x\Big]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\Big[x\Big]_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right] = 0$$

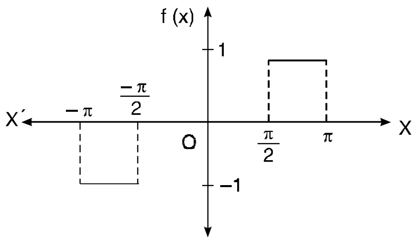
$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nx\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} = -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right]+\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx$$



$$+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2}-\frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]-\frac{1}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]$$

$$b_1 = \frac{2}{\pi},\ b_2 = -\frac{2}{\pi},\ b_3 = \frac{2}{3\pi}$$

Putting the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{1}{\pi}\left[2\sin x-2\sin 2x+\frac{2}{3}\sin 3x+\ldots\right] \qquad \textbf{Ans.}$$

Example 4. *Find the Fourier series for the periodic function*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$f(x+2\pi) = f(x)$$

Solution. Let $f(x) = \dfrac{a_0}{2}+a_1\cos x+a_0\cos 2x+\ldots+b_1\sin x+b_2\sin 2x+\ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}0\,.\,dx+\frac{1}{\pi}\int_0^{\pi}x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}x\cos nx\,dx = \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right] = -\frac{2}{n^2\pi},\quad \text{when } n \text{ is odd}$$

$$= 0,\ \text{when } n \text{ is even.}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = -\frac{(-1)^n}{n}$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ in (1), we get

$$f(x) = \frac{\pi}{4}-\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}\ldots\right]+\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\right] \qquad \textbf{Ans.}$$

DISCONTINUOUS FUNCTIONS

At a point of discontinuity, Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

At $x = c$, $f(x) = \dfrac{1}{2}\,[f(c-0)+f(c+0)]$

Example 5. *Find the Fourier series for $f(x)$, if $f(x) = \begin{cases} -\pi & -\pi<x<0 \\ x, & 0<x<\pi \end{cases}$.*

Deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots = \dfrac{\pi^2}{8}$ (*Warangal, 1996*)

Solution. Let $f(x) = \dfrac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots+a_n\cos nx+\ldots$

$$+\ b_1\sin x+b_2\sin 2x+\ldots+b_n\sin nx+\ldots \qquad \ldots(1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^\pi x\,dx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^0 + (x^2/2)_0^\pi\right] = \frac{1}{\pi}(-\pi^2+\pi^2/2) = -\frac{\pi}{2}$;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^\pi x\cos nx\,dx\right] = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right)_0^\pi\right]$$

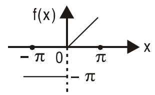
$$= \frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^\pi x\sin nx\,dx\right] = \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1-2\cos n\pi)$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right) + 3\sin x - \frac{\sin 2x}{2}+\frac{3\sin 3x}{3}-\frac{\sin 4x}{4} \quad \ldots(2)$$

Putting $x=0$ in (2), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\infty\right)$ (3)



Now $f(x)$ is discontinuous at $x=0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$ $\therefore$ $f(0) = \frac{1}{2}[f(0-0)+f(0+0)] = -\pi/2$

From (3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]$ or $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ **Proved**

Example 6. *Find the Fourier series expansion of the periodic function of period 2π, defined by*

$f(x) = x$, *if* $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\quad f(x) = \pi - x$, *if* $\frac{\pi}{2} < x < \frac{3\pi}{2}$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi-x)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8}-\frac{\pi^2}{8}\right)+\frac{1}{\pi}\left(\frac{3\pi^2}{2}-\frac{9\pi^2}{8}-\frac{\pi^2}{2}+\frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2}-\frac{9}{8}-\frac{1}{2}+\frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi-x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi-x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}+\frac{\cos\frac{n\pi}{2}}{n^2}-\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}-\frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$+\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n}-\frac{\cos\frac{3n\pi}{2}}{n^2}-\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}+\cos\frac{\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2}+\sin\frac{n\pi}{2}\right)-\frac{1}{n^2}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2}+\frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi-x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{1}{\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]+\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{3\sin\frac{n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos 3\frac{n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right] = \frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2}-\sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ we get

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\right]$$ **Ans.**

Example 7. *Find the Fourier series of the function defined as*

$$f(x) = \begin{cases} x+\pi & \text{for } 0\le x\le\pi \\ -x-\pi & -\pi\le x<0 \end{cases} \quad \textit{and} \quad f(x+2\pi) = f(x).$$

Solution. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_0^\pi f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\,dx + \frac{1}{\pi}\int_0^\pi (x+\pi)\,dx = \frac{1}{\pi}\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2}+\pi x\right)_0^\pi$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}-\pi^2\right)+\frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi^2\right) = \pi\left(\frac{1}{2}-1\right)+\pi\left(\frac{1}{2}+1\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x+\pi)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n} - (-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n} - (1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^n - 1] = \frac{-4}{n^2\pi} \quad \text{if } n \text{ is odd.}$$

$$= 0 \quad \text{if } n \text{ is even.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x+\pi)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}[(1) - 2(-1)^n + (1)] = \frac{2}{n}[1 - (-1)^n]$$

$$= \frac{4}{n}, \qquad \text{if } n \text{ is odd.}$$

$$= 0, \qquad \text{if } n \text{ is even.}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\right)$$

Ans.

Exercise 12.2

1. Find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0 \\ \ \ 1 & \text{for} \quad 0 < x < \pi \end{cases}$$

where $f(x + 2\pi) = f(x)$.

Ans. $\frac{4}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\right]$

2. Find the Fourier series for the function

$$f(x) = \begin{cases} -\dfrac{\pi}{4} & \text{for} \quad -\pi < x < 0 \\ \ \ \dfrac{\pi}{4} & \text{for} \quad 0 < x < \pi \end{cases}$$

and $f(-\pi) = f(0) = f(\pi) = 0$, $f(x) = f(x + 2\pi)$ for all x.

Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Ans. $\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\ldots$

3. Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \le x \le 0 \\ 1 & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ 0 & \text{for} \quad \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

4. Obtain a Fourier series to represent the following periodic function

$$f(x) = 0 \quad \text{when} \quad 0 < x < \pi$$
$$f(x) = 1 \quad \text{when} \quad \pi < x < 2\pi$$

Ans. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

5. Find the Fourier expansion of the function defined in a single period by the relations.

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \pi \\ 2 & \text{for} \quad \pi < x < 2\pi \end{cases}$$

and from it deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Ans. $\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

6. Find a Fourier series to represent the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi < x \le 0 \\ \dfrac{1}{4}\pi x & \text{for} \quad 0 < x < \pi \end{cases}$$

and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Ans. $\frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left(\frac{[(-1)^n - 1]}{4n^2}\cos nx - \frac{(-1)^n\pi}{4n}\sin nx + \ldots\right)$

7. Find the Fourier series for $f(x)$, if

$$f(x) = -\pi \quad \text{for} \quad -\pi < x \le 0$$
$$= x \quad \text{for} \quad 0 < x < \pi$$
$$= \frac{-\pi}{2} \quad \text{for} \quad x = 0$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

(Warangal 1996)

Ans. $-\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + 3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$

8. Obtain a Fourier series to represent the function

$$f(x) = |x| \quad \text{for} \quad -\pi < x < \pi$$

and hence deduce

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

(Madras 1997, Mangalore 1997, A.M.I.E.T.E., Summer 1996)

Ans. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$

9. Expand as a Fourier series, the function $f(x)$ defined as

$$f(x) = \pi + x \quad \text{for} \quad -\pi < x < -\frac{\pi}{2}$$
$$= \frac{\pi}{2} \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$= \pi - x \quad \text{for} \quad \frac{\pi}{2} < x < \pi$$

Ans. $\frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots\right]$

10. Obtain a Fourier series to represent the function

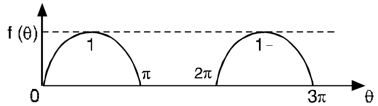
$$f(x) = |\sin x| \quad \text{for} \quad -\pi < x < \pi$$

$\left\{\textbf{Hint}\ \begin{array}{ll} f(x) = -\sin x & \text{for} \quad -\pi < x < 0 \\ \quad\;\; = \sin x & \text{for} \quad 0 < x < \pi \end{array}\right\}$

Ans. $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \ldots\right]$

11. An alternating current after passing through a rectifier has the form

$$i = I\sin\theta \quad \text{for} \quad 0 < \theta < \pi$$
$$= 0 \quad \text{for} \quad \pi < \theta < 2\pi$$



Find the Fourier series of the function.

(*Delhi 1997*) **Ans.** $\frac{I}{\pi} - \frac{2I}{\pi}\left(\frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \ldots\right) + \frac{I}{2}\sin\theta$

12. If $f(x) = 0$ for $-\pi < x < 0$

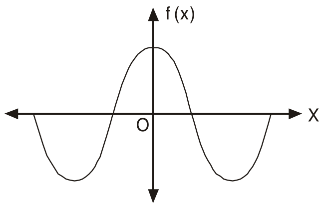
$= \sin x$ for $0 < x < \pi$

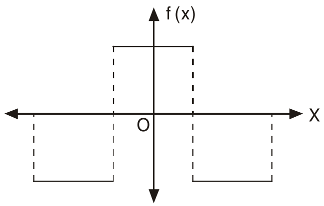
Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2 - 1}$. Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots \infty = \frac{1}{4}(\pi - 2)$

12.8 (a) EVEN FUNCTION

A function $f(x)$ is said to be even (or symmetric) function if, $f(-x) = f(x)$

The graph of such a function is symmetrical with respect to y-axis [$f(x)$ axis]. Here y-axis is a mirror for the reflection of the curve.

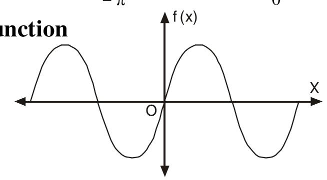


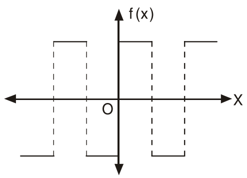


The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \qquad \int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

(b) Odd Function





A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$ to π is zero.

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

As $f(x)$ and $\cos nx$ are both even functions .

$\therefore$ The product of $f(x).\cos nx$ is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

As $\sin nx$ is an odd function so $f(x).\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cosine terms.

Expansion of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

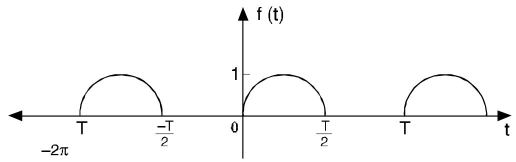
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad [f(x).\cos nx \text{ is odd function.}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

[$f(x).\sin nx$ is even function.]

The series of the odd function will contain only sine terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms



Example 8. *Find the Fourier series expansion of the periodic function of period 2π*

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

(*A.M.I.E.T.E., Winter 1996, Madras 1997, Mangalore 1997, Warangal 1996*)

Solution. $\quad f(x) = x^2, \quad -\pi \le x \le \pi$

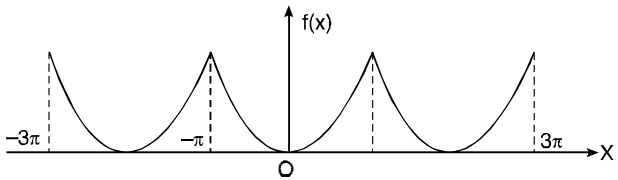
This is an even function. $\therefore \; b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is $\quad f(x) = \dfrac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + a_n\cos nx + \ldots$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\right]$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots = \frac{\pi^2}{12}$$ **Ans.**

Example 9. *Obtain a Fourier expression for*

$$f(x) = x^3 \quad \text{for} \quad -\pi < x < \pi.$$

Solution. $f(x) = x^3$ is an odd function.

$\therefore \qquad a_0 = 0 \quad \text{and} \quad a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3\sin nx\,dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right] = 2.(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right]$$ **Ans.**

12.9 HALF-RANGE SERIES, PERIOD 0 TO π

The given function is defined in the interval $(0, \pi)$ and it is immaterial whatever the function may be outside the interval $(0, \pi)$. To get the series of cosines only we assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$.

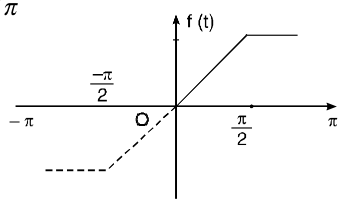
$$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0}$$

To expand $f(x)$ as a sine series we extend the function in the interval $(-\pi, \pi)$ as an odd function.

$$\boldsymbol{b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad \text{and} \quad a_n = 0}$$

Example 10. *Represent the following function by a Fourier sine series :*

$$f(t) = \begin{cases} t, & 0 < t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t \le \pi \end{cases}$$



Solution. $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(t)\sin nt\,dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\sin nt\,dt$$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right) - (1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \ldots$$ **Ans.**

Example 11. *Find the Fourier sine series for the function*

$$f(x) = e^{ax} \quad \text{for} \quad 0 < x < \pi$$

where a is constant.

Solution.

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{e^{ax}}{a^2+n^2} (a \sin nx - n \cos nx) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{e^{a\pi}}{a^2+n^2} (a \sin n\pi - n \cos n\pi) + \frac{n}{a^2+n^2} \right]$$

$$= \frac{2}{\pi} \frac{n}{a^2+n^2} \left[-(-1)^n e^{a\pi} + 1 \right] = \frac{2n}{(a^2+n^2)\pi} \left[1 - (-1)^n e^{a\pi} \right]$$

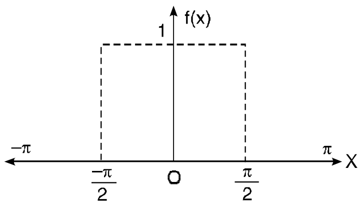
$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \quad b_2 = \frac{2 \,.\, 2 \,.\, (1-e^{a\pi})}{.\, (a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi} \left[\frac{1+e^{a\pi}}{a^2+1^2} \sin x + \frac{2(1-e^{a\pi})}{a^2+2^2} \sin 2x + \ldots \right] \qquad \textbf{Ans.}$$

Exercise 12.3

1. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ 0 & \text{for} \quad \dfrac{\pi}{2} < x < \pi. \end{cases}$$



Ans. $\frac{1}{2} + \frac{2}{\pi} \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \ldots \right]$

2. Find a series of cosine of multiples of x which will represent $f(x)$ in $(0, \pi)$ where

$$f(x) = 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

$$f(x) = \frac{\pi}{2} \quad \text{for} \quad \frac{\pi}{2} < x < \pi$$

Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty = \frac{\pi}{4}$

Ans. $\frac{\pi}{4} - \cos x + \frac{1}{3} \cos 3x - \frac{1}{5} \cos 5x + \ldots$

3. Express $f(x) = x$ as a sine series in $0 < x < \pi$.

Ans. $2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \ldots \right]$

4. Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

Ans. $\frac{\pi}{2} + \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \right)$

5. If $f(x) = x, \quad \text{for} \quad 0 < x < \frac{\pi}{2}$

$= \pi - x, \quad \text{for} \quad \frac{\pi}{2} < x < \pi$

Show that:

(*i*) $f(x) = \frac{4}{\pi} \left(\sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \ldots \right)$ (*Madras 1998, Mysore 1997, Rewa 1994*)

(*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left(\frac{1}{1^2} \cos 2x + \frac{1}{3^2} \cos 6x + \frac{1}{5^2} \cos 10x + \ldots \right)$ (*Delhi 1997, Patel 1997*)

6. Obtain the half-range cosine series for $f(x) = x^2$ in $0 < x < \pi$.

Ans. $\frac{\pi^2}{3} - \frac{4}{\pi} \left(\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \ldots \right)$

7. Find (*i*) sine series and (*ii*) cosine series for the function

$f(x) = e^x \quad \text{for} \quad 0 < x < \pi.$

Ans. (*i*) $\frac{2}{\pi} \sum_{1}^{\infty} n \left[\frac{1-(-1)^n e^{\pi}}{n^2+1} \right] \sin nx$ (*ii*) $\frac{e^{\pi}-1}{\pi} - \frac{2}{\pi} \sum_{1}^{\infty} \frac{1-(-1)^n e^{\pi}}{n^2+1} \cos nx$

8. If $f(x) = x + 1$, for $0 < x < \pi$, find its Fourier (*i*) sine series (*ii*) cosine series. Hence deduce that

(*i*) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$ (*ii*) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$

Ans. (*i*) $\frac{2}{\pi} \left[(\pi+2) \sin x - \frac{\pi}{2} \sin 2x + \frac{1}{3} (\pi+2) \sin 3x - \frac{\pi}{4} \sin 4x + \ldots \right]$

(*ii*) $\frac{\pi}{2} + 1 - 4 \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \right)$

9. Find the Fourier series expansion of the function

$$f(x) = \cos(sx), \quad -\pi \le x \le \pi$$

where s is a fraction. Hence, show that $\cot \theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \ldots.$

(*A.M.I.E.T.E., Summer 1997*)

Ans. $\frac{\sin \pi x}{\pi s} + \frac{1}{\pi} \Sigma \left(\frac{\sin(s\pi + n\pi)}{s+n} + \frac{\sin(s\pi - n\pi)}{s-n} \right) \cos nx$

12.10 CHANGE OF INTERVAL AND FUNCTIONS HAVING ARBITRARY PERIOD

In electrical engineering problems, the period of the function is not always 2π but T or $2c$. This period must be converted to the length 2π. The independent variable x is also to be changed proportionally.

Let the function $f(x)$ be defined in the interval $(-c, c)$. Now we want to change the function to the period of 2π so that we can use the formulae of a_n, b_n as discussed in article 12.6.

$\because$ $2c$ is the interval for the variable x.

$\therefore$ 1 is the interval for the variable $= \frac{x}{2c}$

$\therefore$ 2π is the interval for the variable $= \frac{x \, 2\pi}{2c} = \frac{\pi x}{c}$

so put $\quad z = \frac{\pi x}{c} \quad \text{or} \quad x = \frac{zc}{\pi}$

Thus the function $f(x)$ of period $2c$ is transformed to the function

$f\left(\frac{cz}{\pi}\right)$ or $F(z)$ of period 2π.

$F(z)$ can be expanded in the Fourier series.

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + a_1 \cos z + a_2 \cos 2z + b_1 \sin z + b_2 \sin 2z + \ldots$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(z)\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right) dz$$

$$= \frac{1}{c}\int_0^{2c} f(x)\, d\left(\frac{\pi x}{c}\right) = \boldsymbol{\frac{1}{c}\int_0^{2c} f(x)dx} \quad \textbf{put} \quad z = \boldsymbol{\frac{\pi x}{c}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(z)\cos nz\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)\cos nx\,dz$$

$$= \frac{1}{\pi}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\, d\left(\frac{\pi x}{c}\right) = \boldsymbol{\frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx.} \qquad \left[\text{Put } z = \frac{\pi x}{c}\right]$$

Similarly, $$\boldsymbol{b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx.}$$

Cor. Half range series [Interval $(0, c)$]

Cosine series:

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + a_n\cos\frac{n\pi x}{c} + \ldots$$

where $$a_0 = \frac{2}{c}\int_0^c f(x)\,dx, \quad a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

Sine series:

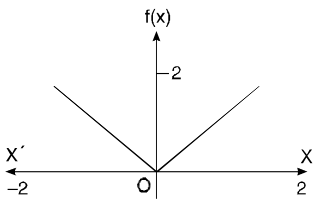
$$f(x) = b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots + b_n\sin\frac{n\pi x}{c} + \ldots$$

where $$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx.$$

Example 12. *A periodic function of period 4 is defined as*

$$f(x) = |x|, \quad -2 < x < 2.$$

Find its Fourier series expansion.



Solution.

$$f(x) = |x| \qquad -2 < x < 2$$
$$f(x) = x \qquad 0 < x < 2$$
$$= -x \qquad -2 < x < 0$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

$$= -\frac{8}{n^2\pi^2} \qquad \text{if } n \text{ is odd.}$$
$$= 0 \qquad \text{if } n \text{ is even}$$

$b_n = 0$ as $f(x)$ is even function.

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots$$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \ldots\right] \quad \textbf{Ans.}$$

Example 13. *Find Fourier half-range even expansion of the function,*

$$f(x) = (-x/l) + 1, \quad 0 \le x \le l.$$

Solution. $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l\left(-\frac{x}{l}+1\right)dx$$

$$= \frac{2}{l}\left[-\frac{x^2}{2l} + x\right]_0^l = \frac{2}{l}\left[-\frac{l^2}{2l} + l\right] = \frac{2l}{l}\left[-\frac{1}{2}+1\right] = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l\left(-\frac{x}{l}+1\right)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(-\frac{x}{l}+1\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \left(-\frac{1}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \frac{l}{n^2\pi^2}\cos n\pi + \frac{l}{n^2\pi^2}\right] = \frac{2}{l}\frac{l}{n^2\pi^2}[-(-1)^n + 1] = \frac{2}{n^2\pi^2}[1 - (-1)^n]$$

$$= \frac{4}{n^2\pi^2} \quad \text{when } n \text{ is odd.}$$
$$= 0 \quad \text{when } n \text{ is even.}$$

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l}\ldots\right] \quad \textbf{Ans.}$$

Example 14. *Find the Fourier half-range cosine series of the function*

$$f(t) = 2t, \qquad 0 < t < 1$$
$$= 2(2-t), \qquad 1 < t < 2$$

(Kuvempu 1996, A.M.I.E.T.E.,Summer 1997 1996)

Solution.

$$f(t) = 2t, \qquad 0 < t < 1$$
$$= 2(2-t), \qquad 1 < t < 2$$

Let $$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{c} + a_2 \cos\frac{2\pi t}{c} + a_3 \cos\frac{3\pi t}{c} + \ldots$$
$$+ b_1 \sin\frac{\pi t}{c} + b_2 \sin\frac{2\pi t}{c} + b_3 \sin\frac{3\pi t}{c} + \ldots \qquad \ldots(1)$$

Here $c = 2$, because it is half range series.

Hence $$a_0 = \frac{2}{c}\int_0^c f(t)\,dt = \frac{2}{2}\int_0^1 2t\,dt + \frac{2}{2}\int_1^2 2(2-t)\,dt$$

$$= \left[t^2\right]_0^1 + \left[2\left(2t - \frac{t^2}{2}\right)\right]_1^2 = 1 + \left[(4t - t^2)\right]_1^2 = 1 + (8 - 4 - 4 + 1) = 2$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{c}\,dt = \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \frac{2}{2}\int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \left[2t\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_0^1$$
$$+ \left[(4-2t)\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (-2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_1^2$$

$$= \left[\frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\right] + \left[0 - \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1 - \frac{n\pi}{2}\sin\frac{n\pi}{2}\right]$$

If $n = 1$, $$a_1 = \frac{8}{\pi^2}\left[0 - 1 - \frac{\pi}{2}\right] = -\frac{8}{\pi^2} - \frac{4}{\pi}.$$

If $n = 2$, $$a_2 = \frac{8}{4\pi^2}[-1-1] = -\frac{16}{4\pi^2} = -\frac{4}{\pi^2}$$

If $n = 3$, $$a_3 = \frac{8}{9\pi^2}\left[0 - 1 + \frac{3\pi}{2}\right] = -\frac{8}{9\pi^2} + \frac{4}{3\pi}$$

Putting the values of $a_0, a_1, a_2, a_3 \ldots$ in (1) we get

$$f(t) = 1 - \left(\frac{8}{\pi^2} + \frac{4}{\pi}\right)\cos\frac{\pi t}{2} - \frac{4}{\pi^2}\cos\frac{2\pi t}{2} + \left(-\frac{8}{9\pi^2} + \frac{4}{3\pi}\right)\cos\frac{3\pi t}{2} + \ldots \quad \textbf{Ans.}$$

Example 15. *Obtain the Fourier cosine series expansion of the periodic function defined by*

$$f(t) = \sin\left(\frac{\pi t}{l}\right), \quad 0 < t < l$$

Solution. $$f(t) = \sin\left(\frac{\pi t}{l}\right), \quad 0 < t < l$$

$$a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)dt = \frac{2}{l}\left(-\frac{l}{\pi}\cos\frac{\pi t}{l}\right)_0^l = -\frac{2}{\pi}(\cos\pi - \cos 0) = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)\cos\frac{n\pi t}{l}\,dt = \frac{1}{l}\int_0^l \left[\sin\left(\frac{\pi t}{l} + \frac{n\pi t}{l}\right) - \sin\left(\frac{n\pi t}{l} - \frac{\pi t}{l}\right)\right]dt$$

$$= \frac{1}{l}\int_0^l \sin(n+1)\frac{\pi t}{l}\,dt - \frac{1}{l}\int_0^l \sin(n-1)\frac{\pi t}{l}\,dt$$

$$= \frac{1}{l}\left[-\frac{l}{(n+1)\pi}\cos\frac{(n+1)\pi t}{l}\right]_0^l - \frac{1}{l}\left[\frac{l}{(n-1)\pi}\cos\frac{(n-1)\pi t}{l}\right]_0^l$$

$$= \frac{-1}{(n+1)\pi}[\cos(n+1)\pi - \cos 0] + \frac{1}{(n-1)\pi}[\cos(n-1)\pi - \cos 0]$$

$$= \frac{1}{(n+1)\pi}[(-1)^{n+1} - 1] + \frac{1}{(n-1)\pi}[(-1)^{n+1} - 1]$$

$$= (-1)^{n+1}\left[-\frac{1}{(n+1)\pi} + \frac{1}{(n-1)\pi}\right] + \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi}$$

$$= (-1)^{n+1}\frac{2}{(n^2-1)\pi} - \frac{2}{(n^2-1)\pi} = \frac{2}{(n^2-1)\pi}[(-1)^{n+1} - 1]$$

$$= \frac{-4}{(n^2-1)\pi} \quad \text{when } n \text{ is even}$$

$$= 0 \quad \text{when } n \text{ is odd.}$$

The above formula for finding the value of a_1 is not applicable.

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi t}{l}\cos\frac{\pi t}{l}\,dt = \frac{1}{l}\int_0^l \sin\frac{2\pi t}{l}\,dt$$

$$= \frac{1}{l}\left(-\frac{l}{2\pi}\cos\frac{2\pi t}{l}\right)_0^l = -\frac{1}{2\pi l}(\cos 2\pi - \cos 0) = 0 = \frac{1}{2\pi l}(1-1) = 0$$

$$f(t) = \frac{a_0}{2} + a_1\cos\frac{\pi t}{l} + a_2\cos\frac{2\pi t}{l} + a_3\cos\frac{3\pi t}{l} + a_4\cos\frac{4\pi t}{l} + \ldots$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi t}{l} + \frac{1}{15}\cos\frac{4\pi t}{l} + \frac{1}{35}\cos\frac{6\pi t}{l} + \ldots\right] \qquad \textbf{Ans.}$$

Example 16. *Find the Fourier series expansion of the periodic function of period 1*

$$f(x) = \frac{1}{2} + x, \quad -\frac{1}{2} < x \le 0$$
$$= \frac{1}{2} - x, \quad 0 < x < \frac{1}{2}$$

(A.M.I.E.T.E., Winter 1996)

Solution. Let $$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots$$
$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots \qquad \ldots(1)$$

Here $2c = 1$ or $c = \frac{1}{2}$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2} + x\right)dx + \frac{1}{1/2}\int_0^{1/2}\left(\frac{1}{2} - x\right)dx$$

$$= 2\left[\frac{x}{2} + \frac{x^2}{2}\right]_{-1/2}^{0} + 2\left[\frac{x}{2} - \frac{x^2}{2}\right]_0^{1/2} = 2\left[\frac{1}{4} - \frac{1}{8}\right] + 2\left[\frac{1}{4} - \frac{1}{8}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos\frac{n\pi x}{1/2}\,dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\frac{\sin 2n\pi x}{2n\pi} - (1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+ 2\left[\left(\frac{1}{2}-x\right)\frac{\sin 2n\pi x}{2n\pi} - (-1)\left(\frac{-\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[0+\frac{1}{4n^2\pi^2}-\frac{(-1)^n}{4n^2\pi^2}\right] + 2\left[0-\frac{(-1)^n}{4n^2\pi^2}+\frac{1}{4n^2\pi^2}\right] = \frac{1}{\pi^2}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{n^2\pi^2} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin\frac{n\pi x}{1/2}\,dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right) - (1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+ 2\left[\left(\frac{1}{2}-x\right)\left(-\frac{\cos 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[-\frac{1}{4n\pi}\right] + 2\left[\frac{1}{4n\pi}\right] = 0$$

Substituting the values of $a_0, a_1, a_2, a_3, \ldots b_1, b_2, b_3 \ldots$ in (1) we have

$$f(x) = \frac{1}{4} + \frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2} + \frac{\cos 6\pi x}{3^2} + \frac{\cos 10\pi x}{5^2} + \ldots\right] \quad \textbf{Ans.}$$

Example 17. *Prove that*

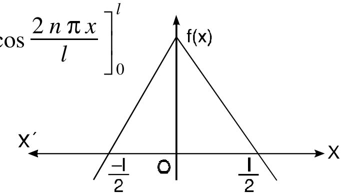
$$\frac{l}{2} - x = \frac{l}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}, \quad 0 < x < l.$$

Solution. $\qquad f(x) = \dfrac{l}{2} - x$

$$a_0 = \frac{1}{l/2}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l\left(\frac{l}{2}-x\right)dx = \frac{2}{l}\left[\frac{lx}{2}-\frac{x^2}{2}\right]_0^l = 0$$

$$a_n = \frac{1}{l/2}\int_0^l f(x)\cos\frac{n\pi x}{l/2}\,dx = \frac{2}{l}\int_0^l\left(\frac{l}{2}-x\right)\cos\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{l}{2}-x\right)\frac{l}{2n\pi}\sin\frac{2n\pi x}{l} + (-1)\frac{l^2}{4n^2\pi^2}\cos\frac{2n\pi x}{l}\right]_0^l$$



$$= \frac{2}{l}\left[0 - \frac{l^2}{4n^2\pi^2}\cos 2n\pi + \frac{l^2}{4n^2\pi^2}\right]$$

$$= \frac{2}{l}\frac{l^2}{4n^2\pi^2}(-\cos 2n\pi + 1) = \frac{l}{2n^2\pi^2}(-1+1) = 0$$

$$b_n = \frac{1}{l/2}\int_0^l f(x)\sin\frac{n\pi x}{l/2}\,dx = \frac{2}{l}\int_0^l\left(\frac{l}{2}-x\right)\sin\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{l}{2}-x\right)\left(-\frac{l}{2n\pi}\cos\frac{2n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{4n^2\pi^2}\sin\frac{2n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l}{2}\frac{l}{2n\pi}\cos 2n\pi - 0 + \frac{l}{2}\cdot\frac{l}{2n\pi}(1)\right] = \frac{2}{l}\left[\frac{l^2}{2n\pi}\right] = \frac{l}{n\pi}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{n\pi x}{l/2} + a_2\cos\frac{2n\pi x}{l/2} + a_3\cos\frac{3n\pi x}{l/2} + \ldots$$

$$+ b_1\sin\frac{n\pi x}{l/2} + b_2\sin\frac{2n\pi x}{l/2} + b_3\sin\frac{3n\pi x}{l/2} + \ldots$$

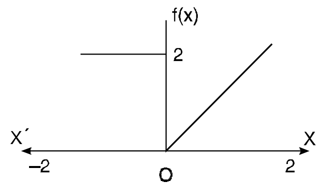
$$\frac{l}{2} - x = \frac{l}{\pi}\sin\frac{2\pi x}{l} + \frac{l}{2\pi}\sin\frac{4\pi x}{l} + \frac{l}{3\pi}\sin\frac{6\pi x}{l} + \ldots$$

$$= \frac{l}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l} \qquad \textbf{Proved.}$$

Example 18. *Find the Fourier series corresponding to the function f (x) defined in* $(-2, 2)$ *as follows*

$$f(x) = 2 \quad \text{in} \quad -2 \le x \le 0$$
$$= x \quad \text{in} \quad 0 < x < 2.$$

Solution. Here the interval is $(-2, 2)$ and $c = 2$



$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2.\,dx + \int_0^2 x.\,dx\right]$$

$$= \frac{1}{2}\left[[2x]_{-2}^{0} + \left(\frac{x^2}{2}\right)_0^2\right] = \frac{1}{2}[4+2] = 3$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left(\sin\frac{n\pi x}{2}\right)_{-2}^{0} + \left(x\frac{2}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$= -\frac{4}{n^2\pi^2} \qquad \text{when } n \text{ is odd}$$

$$= 0 \qquad \text{when } n \text{ is even.}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + (1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right] = \frac{1}{2}\left[-\frac{4}{n\pi}\right] = -\frac{2}{n\pi}$$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + \ldots$$

$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + b_3\sin\frac{3\pi x}{c} + \ldots$$

$$= \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\right\}$$

$$-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\right\} \quad \textbf{Ans.}$$

Example 19. *Expand $f(x) = e^x$ in a cosine series over (0, 1).*

Solution. $f(x) = e^x$ and $c = 1$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = 2(e-1)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}\,dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi x + \cos n\pi x)\right]_0^1 = 2\left[\frac{e}{n^2\pi^2+1}(n\pi\sin n\pi + \cos n\pi) - \frac{1}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}[(-1)^n e - 1]$$

$$f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + \ldots\ldots\ldots$$

$$e^x = e - 1 + 2\left[\frac{-e-1}{\pi^2+1}\cos\pi x + \frac{e-1}{4\pi^2+1}\cos 2\pi x + \frac{-e-1}{9\pi^2+1}\cos 3\pi x + \ldots\right] \quad \textbf{Ans.}$$

Exercise 12.4

1. Find the Fourier series to represent $f(x)$, where

$$f(x) = -a \qquad -c < x < 0$$
$$= a \qquad 0 < x < c$$

Ans. $\frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right]$

2 Find the half-range sine series for the function

$$f(x) = 2x - 1 \qquad 0 < x < 1.$$

Ans. $-\frac{2}{\pi}\left[\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \ldots\right]$

3. Express $f(x) = x$ as a cosine, half range series in $0 < x < 2$.

Ans. $1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]$

4. Find the Fourier series of the function

$$f(x) = \begin{bmatrix} -2 & \text{for} & -4 < x < -2 \\ x & \text{for} & -2 < x < 2 \\ 2 & \text{for} & 2 < x < 4 \end{bmatrix}$$

Ans. $\frac{4}{\pi} + \frac{8}{\pi^2}\sin\frac{\pi x}{4} - \frac{2}{\pi}\sin\frac{2\pi x}{4} + \left(\frac{4}{3\pi} - \frac{8}{3^2\pi}\right)\sin\frac{3\pi x}{4} - \frac{2}{2\pi}\sin\frac{4\pi x}{4} + \ldots$

5. Find the Fourier series to represent

$$f(x) = x^2 - 2 \qquad \text{from} \qquad -2 < x < 2.$$

Ans. $-\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2^2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2}\ldots\right]$

6. If $f(x) = e^{-x}$ $\quad -c < x < c$, show that

$$f(x) = (e^c - e^{-c})\left\{\frac{1}{2c} - c\left(\frac{1}{c^2+\pi^2}\cos\frac{\pi x}{c} - \frac{1}{c^2+4\pi^2}\cos\frac{2\pi x}{c} + \ldots\right)\right.$$
$$\left. -\pi\left(\frac{1}{c^2+\pi^2}\sin\frac{\pi x}{c} - \frac{2}{c^2+4\pi^2}\sin\frac{2\pi x}{c}\ldots\right)\right\}$$

(*Hamirpur 1996, Mysore 1994*)

7. A sinusodial voltage E sin ωt is passed through a half wave rectifier which clips the negative portion of the wave. Develop the resulting portion of the function

$$u(t) = 0 \qquad \text{when } -\frac{T}{2} < t < 0$$

$$= E\sin\omega t \qquad \text{when } 0 < t < \frac{T}{2} \qquad \left(T = \frac{2\pi}{\omega}\right)$$

(*Mangalore 1997*)

Ans. $\frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \frac{1}{5.7}\cos 6\omega t + \ldots\right]$

8. A periodic square wave has a period 4. The function generating the square is

$$f(t) = 0 \qquad \text{for} \quad -2 < t < -1$$
$$= k \qquad \text{for} \quad -1 < t < 1$$
$$= 0 \qquad \text{for} \quad 1 < t < 2$$

Find the Fourier series of the function.

Ans. $f(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \ldots\right]$

9. Find a Fourier series to represent x^2 in the interval $(-l, l)$. (*Nagpur 1997*)

Ans. $\frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\cos\pi x - \frac{\cos\pi x}{2^2} + \frac{\cos 3\pi x}{3^2}\right]$

12.11. PARSEVAL'S FORMULA

$$\int_{-c}^{c}[f(x)]^2\,dx = c\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right\}$$

Solution. We know that $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right) \qquad \ldots (1)$

Multiplying (1) by $f(x)$, we get

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n\pi x}{c} \quad ...(2)$$

Integrating term by term from $-c$ to c, we have

$$\int_{-c}^{c} [f(x)]^2\, dx = \frac{a_0}{2} \int_{-c}^{c} f(x)\, dx + \sum_{n=1}^{\infty} a_n \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c}\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c}\, dx \quad ...(3)$$

In article 12.10, we have the following results

$$\int_{-c}^{c} f(x)\, dx = c\, a_0$$

$$\int_{-c}^{c} f(x) \cos \frac{n\pi x}{c}\, dx = c\, a_n$$

$$\int_{-c}^{c} f(x) \sin \frac{n\pi x}{c}\, dx = c\, b_n$$

On putting these integrals in (3), we get

$$\int_{-c}^{c} [f(x)]^2\, dx = c \frac{a_0^2}{2} + \sum_{n=1}^{\infty} c\, a_n^2 + \Sigma\, c\, b_n^2 = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

This is the Parseval's formula

Note. 1. If $0 < x < 2c$, then $\int_0^{2c} [f(x)]^2\, dx = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$

2. If $0 < x < c$ (Half range cosine series), $\int_0^{c} [f(x)]^2 = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

3. If $0 < x < c$ (Half range sine series), $\int_0^{c} [f(x)]^2\, dx = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} b_n^2\right]$

4. R.M.S. $= \left\{\dfrac{\int_a^b [f(x)]^2\, dx}{b-a}\right\}^{\frac{1}{2}}$

Example 20. *By using the sine series for $f(x) = 1$ in $0 < x < \pi$ show that*

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$ *(Hamirpur 1996)*

Solution. sine series is $f(x) = \Sigma\, b_n \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (1) \sin nx\, dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}\,[\cos n\pi - 1] = \frac{-2}{n\pi}\,[(-1)^n - 1]$$

$$= \frac{4}{n\pi} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

Then, the sine series is

$$1 = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \frac{4}{7\pi} \sin 7x + ...$$

$$\int_0^{c} [f(x)]^2\, dx = \frac{c}{2}\left[b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + ...\right]$$

$$\int_0^{\pi} (1)^2\, dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + ...\right]$$

$$[x]_0^{\pi} = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$ **Proved.**

Example 21. *If* $f(x) = \begin{cases} \pi x, & o < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$

using half range cosine series, show that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\pi^4}{96}$$

Solution. Half range cosine series is

$$f(x) = \frac{a_0}{2} + \Sigma\, a_n \cos \frac{n\pi x}{c}$$

where $$a_0 = \frac{2}{c} \int_0^{c} f(x)\, dx = \frac{2}{2}\left[\int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)\, dx\right]$$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \frac{\pi}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \pi$$

$$a_n = \frac{2}{c} \int_0^{c} f(x) \cos \frac{n\pi x}{c}\, dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x \cos \frac{n\pi x}{2}\, dx + \int_1^2 \pi(2-\pi) \cos \frac{n\pi x}{2}\, dx\right]$$

$$= \pi\left[x\,\frac{\sin \frac{\pi x}{2}}{\frac{n\pi}{2}} - \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \pi\left[(2-x)\,\frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right] + \pi\left[0 - \frac{4}{n^2\pi^2} \cos n\pi - \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2}\right]$$

$$= \pi\left[\frac{8}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2} \cos n\pi\right] = \frac{4}{n^2\pi}\left[2 \cos \frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$a_1 = 0,\ a_2 = \frac{-4}{\pi},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{-4}{9\pi} \ldots\ldots$$

$$\int_0^c [f(x)]^2\,dx = \frac{c}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\right]$$

$$\int_0^1 (\pi x)^2\,dx + \int_1^2 \pi^2 (2-x)^2\,dx = \frac{2}{2}\left[\frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots\ldots\right]$$

$$\pi^2\left[\frac{x^3}{3}\right]_0^1 - \pi^2\left[\frac{(2-x)^3}{3}\right]_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots\ldots$$

$$\frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[1 + \frac{1}{81} + \ldots\ldots\right]$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots$$

Ans.

Example 22. *Prove that for $0 < x < \pi$*

$$(a)\quad x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right]$$

$$(b)\quad x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots\ldots\right]$$

Deduce from (a) and (b) respectively that

$$(c)\ \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \qquad (d)\ \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi}{945}$$

Solution. Half range cosine series

$$a_0 = \frac{2}{\pi}\int_0^\pi x(\pi - x) = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\,\frac{\sin nx}{n} - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 - \frac{\pi(-1)^n}{n^2} + 0 - \frac{\pi}{n^2}\right] = \frac{2}{\pi}\left(\frac{\pi}{n^2}\right)\left[-(-1)^n - 1\right]$$

$$= \frac{-4}{n^2} \qquad \text{when } n \text{ is even}$$

$$= 0 \qquad \text{when } n \text{ is odd}$$

Hence,

$$x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \ldots\ldots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^\pi x^2(\pi - x)^2\,dx = \frac{a_0^2}{2} + \Sigma\ a_n^2$$

$$\frac{2}{\pi}\int_0^\pi (\pi^2 x^2 - 2\pi x^3 + x^4)\,dx = \frac{1}{2}\left(\frac{\pi^4}{9}\right) + 16\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2 x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^\pi = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{3} - \frac{2\pi^5}{4} + \frac{\pi^5}{5}\right] = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\right]$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18} + \sum_{n=1}^{\infty}\frac{1}{n^4} \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

(*b*) Half range sine series

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\,\frac{\cos nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-2\,\frac{(-1)^n}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left[-(-1)^n + 1\right]$$

$$= \frac{8}{n^3\pi} \qquad \text{when } n \text{ is odd}$$

$$= 0 \qquad \text{when } n \text{ is even}$$

$$\therefore \quad x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\ldots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^\pi x^2(\pi - x)^2\,dx = \Sigma\ b_n^2$$

$$\frac{\pi^2}{15} = \frac{64}{\pi^2}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\ldots\right]$$

$$\frac{\pi^4}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6}$$

$$\text{Let } S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots\ldots = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\ldots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\ldots\right)$$

$$S = \frac{\pi^4}{960} + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\right) = \frac{\pi^4}{960} + \frac{1}{2^6}\left[\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots\right]$$

$$S = \frac{\pi^4}{960} + \frac{S}{64}$$

$$S - \frac{S}{64} = \frac{\pi^4}{960} \quad \text{or} \quad \frac{63s}{64} = \frac{\pi^4}{960}$$

$$S = \frac{\pi^4}{960} \times \frac{64}{63} = \frac{\pi^4}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^4}{945}$$ **Proved.**

Exercise 12.5

1. Prove that in $0 < x < c$,

$$x = \frac{c}{2} - \frac{4c}{\pi^2}\left(\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \frac{1}{5^2}\cos\frac{5\pi x}{c} + ...\right)$$

and deduce that

(i) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\pi^4}{96}$ (ii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{\pi^4}{90}$

12.12. FOURIER SERIES IN COMPLEX FORM

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + ... + a_n\cos\frac{n\pi x}{l} + ...$$

$$+ b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + ... + b_n\sin\frac{n\pi x}{l} + ... \quad ... (1)$$

We know that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

On putting the values of cos x and sin x in (1), we get

$$f(x) = \frac{a_0}{2} + a_1\frac{e^{\frac{i\pi x}{l}} + e^{\frac{-i\pi x}{l}}}{2} + a_2\frac{e^{\frac{2i\pi x}{l}} + e^{\frac{-2i\pi x}{l}}}{2} + ... + b_1\frac{e^{\frac{i\pi x}{l}} - e^{\frac{i\pi x}{l}}}{2i} + b_2\frac{e^{\frac{2i\pi x}{l}} - e^{\frac{-2i\pi x}{l}}}{2i} + ...$$

$$= \frac{a_0}{2} + \left(a_1 - ib_1\right)e^{\frac{i\pi x}{l}} + \left(a_2 - ib_2\right)e^{\frac{2i\pi x}{l}} + ... + \left(a_1 + ib_1\right)e^{\frac{-i\pi x}{l}} + \left(a_2 + ib_2\right)e^{\frac{-2i\pi x}{l}} + ...$$

$$= c_0 + c_1 e^{\frac{i\pi x}{l}} + c_2 e^{\frac{2i\pi x}{l}} + ... + c_{-1} e^{\frac{-i\pi x}{l}} + c_{-2} e^{\frac{2i\pi x}{l}} + ...$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{l}}$$

$$c_n = \frac{1}{2}\left(a_n - ib_n\right), \quad c_{-n} = \frac{1}{2}\left(a_n + ib_n\right)$$

where $c_0 = \frac{a_0}{2} = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\,dx$

$$c_n = \frac{1}{2}\left[\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right] \Rightarrow c_n = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\left\{\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right\}dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\,e^{\frac{-in\pi x}{l}}\,dx$$

$$c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\,e^{\frac{in\pi x}{l}}\,dx$$

Example 23. *Obtain the complex form of the Fourier series of the function*

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Solution. $c_0 = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot e^{-inx}\,dx + \int_0^{\pi} 1\cdot e^{-inx}\,dx\right] = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$= -\frac{1}{2n\pi i}\left[e^{-in\pi} - 1\right] = -\frac{1}{2n\pi i}\left[\cos n\pi - 1\right] = -\frac{1}{2n\pi i}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \frac{1}{in\pi}, & n \text{ is odd} \\ 0\ , & n \text{ is even} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + ...\right] + \frac{1}{i\pi}\left[\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + ...\right]$$

$$= \frac{1}{2} - \frac{1}{i\pi}\left[\left(e^{ix} - e^{-ix}\right) + \frac{1}{3}\left(e^{3ix} - e^{-3ix}\right) + \frac{1}{5}\left(e^{5ix} - e^{-5ix}\right) + ...\right]$$ **Ans.**

Exercise 12.6

Find the complex form of the Fourier series of

1. $f(x) = e^{-x}, \ -1 \le x \le 1.$ **Ans.** $\sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1 \cdot e^{in\pi x}$

2. $f(x) = e^{ax}, \ -l < x < l$ **Ans.** $\frac{2}{\pi} - \frac{2}{\pi}\left[\frac{e^{2it} + e^{-2it}}{1\cdot 3} + \frac{e^{4it} + e^{-4it}}{3\cdot 5} + \frac{e^{6it} + e^{-6it}}{5 - 7} + ...\right]$

3. $f(x) = \cos ax, \ -\pi < x < \pi$ **Ans.** $\frac{a}{\pi}\sin a\pi + \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{a^2 - n^2}$

12.13 PRACTICAL HARMONIC ANALYSIS

Sometimes the function is not given by a formula, but by a graph or by a table of corresponding values. The process of finding the Fourier series for a function given by such values of the function and independent variable is known as **Harmonic Analysis.** The Fourier constants are evaluated by the following formulae :

(1) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= 2\,\frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\,dx \quad \left[\text{Mean} = \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

or $a_0 = 2$ [mean value of $f(x)$ in $(0, 2\pi)$]

(2) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = 2\,\frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\cos nx\,dx$$

$a_n = 2$ [mean value of $f(x)\cos nx$ in $(0, 2\pi)$]

(3) $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = 2\,\frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\sin nx\,dx$$

$b_n = 2$ [mean value of $f(x)\sin nx$ in $(0, 2\pi)$]

Fundamental of first harmonic. The term ($a_1 \cos x + b_1 \sin x$) in Fourier series is called the fundamental or first harmonic.

Second harmonic. The term ($a_2 \cos 2x + b_2 \sin 2x$) in Fourier series is called the second harmonic and so on.

Example 24. *Find the Fourier series as far as the second harmonic to represent the function given by table below :*

x	0º	30º	60º	90º	120º	150º	180º	210º	240º	270º	300º	330º
$f(x)$	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Solution

x^o	*sin x*	*sin 2x*	*cos x*	*cos 2x*	$f(x)$	$f(x)$. *sin x*	$f(x)$. *sin 2x*	$f(x)$. *cos x*	$f(x)$. *cos 2x*
0º	0	0	1	1	2.34	0	0	2.340	2.340
30º	0.50	0.87	0.87	0.50	3.01	1.505	2.619	2.619	1.505
60º	0.87	0.87	0.50	– 0.50	3.69	3.210	3.210	1.845	– 1.845
90º	1.00	0	0	– 1.00	4.15	4.150	0	0	– 4.150
120º	0.87	– 0.87	– 0.50	– 0.50	3.69	3.210	– 3.210	– 1.845	– 1.845
150º	0.50	– 0.87	– 0.87	0.50	2.20	1.100	– 1.914	– 1.914	1.100
180º	0	0	– 1	1.00	0.83	0	0	– 0.830	0.830
210º	– 0.50	0.87	– 0.87	0.50	0.51	– 0.255	0.444	– 0.444	0.255
240º	– 0.87	0.87	– 0.50	– 0.50	0.88	– 0.766	0.766	– 0.440	– 0.440
270º	– 1.00	0	0	– 1.00	1.09	– 1.090	0	0	– 1.090
300º	– 0.87	– 0.87	0.50	– 0.50	1.19	– 1.035	– 1.035	0.595	– 0.595
330º	– 0.50	– 0.87	0.87	0.50	1.64	– 0.820	– 1.427	1.427	0.820
					25.22	9.209	– 0.547	3.353	– 3.115

$$a_0 = 2 \times \text{Mean of } f(x) = 2 \times \frac{25.22}{12} = 4.203$$

$$a_1 = 2 \times \text{Mean of } f(x) \cos x = 2 \times \frac{3.353}{12} = 0.559$$

$$a_2 = 2 \times \text{Mean of } f(x) \cos 2x = 2 \times \frac{-3.115}{12} = -0.519$$

$$b_1 = 2 \times \text{Mean of } f(x) \sin x = 2 \times \frac{9.209}{12} = 1.535$$

$$b_2 = 2 \times \text{Mean of } f(x) \sin 2x = 2 \times \frac{-0.547}{12} = -0.091$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$$

$$= 2.1015 + 0.559 \cos x - 0.519 \cos 2x + \ldots + 1.535 \sin x - 0.091 \sin 2x + \ldots \quad \textbf{Ans.}$$

Example 25. *A machine completes its cycle of operations every time as certain pulley completes a revolution. The displacement f (x) of a point on a certain portion of the machine is given in the table given below for twelve positions of the pulley, x being the angle in degree turned through by the pulley. Find a Fourier series to represent f (x) for all values of x.*

x	30º	60º	90º	120º	150º	180º	210º	240º	270º	300º	330º	360º
$f(x)$	7.976	8.026	7.204	5.676	3.674	1.764	0.552	0.262	0.904	2.492	4.736	6.824

Solution.

x	*sin x*	*sin 2x*	*sin 3x*	*cos x*	*cos 2x*	*cos 3x*	$f(x)$	$f(x) \times$ *sin x*	$f(x) \times$ *sin 2x*	$f(x) \times$ *sin 3x*	$f(x) \times$ *cos x*	$f(x) \times$ *cos 2x*	$f(x) \times$ *cos 3x*
30º	0.50	087	1	0.87	0.50	0	7.976	3.988	6.939	7.976	6.939	3.988	0
60º	0.87	0	0	0.50	–0.50	–1	8.026	6.983	6.983	0	– 4.013	4.013	–8.026
90º	1.00	0.87	–1	0	–1	0	7.204	7.204	0	–7.204	0	–7.204	0
120º	0.50	–0.87	0	– 0.50	– 0.50	1	5.676	4.938	– 4.939	0	–2.838	–2.838	5.676
150º	0.50	–0.87	1	– 0.87	0.50	0	3.674	1.837	–3.196	–3.196	–3.196	1.837	0
180º	0	0	0	–1	1	–1	1.764	0	0	–1.764	–1.764	1.764	–1.764
210º	–0.50	0.87	–1	– 0.87	0.50	0	0.552	–0.276	0.480	0.480	– 0.480	0.276	0
240º	–0.87	0.87	0	– 0.50	– 0.50	1	0.262	–0.228	0.228	– 0.131	– 0.131	0.131	0.262
270º	–1.00	0	1	0	–1.00	0	0.904	–0.904	0	0	0	– 0.904	0
300º	–0.87	–0.87	0	0.50	–0.50	–1	2.492	–2.168	–2.168	1.246	1.246	–1.296	–2.492
330º	–0.50	–0.87	–1	0.87	0.50	0	4.736	–2.368	– 4120	4.120	4.120	2.368	0
360º	0	0	0	1	1	1	6.824	0	0	0	6.824	6.824	6.824
						Σ	50.09	19.206	0.207	0.062	14.733	0.721	0.460

$$a_0 = 2 \times \text{Mean value of } f(x) = 2 \times \frac{50.09}{12} = 8.34$$

$$a_1 = 2 \times \text{Mean value of } f(x) \cos x = 2 \times \frac{14.733}{12} = 2.45$$

$$a_2 = 2 \times \text{Mean value of } f(x) \cos 2x = 2 \times \frac{0.721}{12} = 0.12$$

$$a_3 = 2 \times \text{Mean value of } f(x) \cos 3x = 2 \times \frac{0.460}{12} = 0.08$$

$$b_1 = 2 \times \text{Mean value of } f(x) \sin x = 2 \times \frac{19.206}{12} = 3.16$$

$$b_2 = 2 \times \text{Mean value of } f(x) \sin 2x = 2 \times \frac{0.207}{12} = 0.03$$

$$b_3 = 2 \times \text{Mean value of } f(x) \sin 3x = 2 \times \frac{0.062}{12} = 0.01$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

$$= 4.17 + 2.45 \cos x + 0.12 \cos 2x + 0.08 \cos 3x + ..$$

$$+ 3.16 \sin x + 0.03 \sin 2x + 0.01 \sin 3x + \ldots$$ **Ans.**

Example 26. *Obtain the constant term and the coefficient of the first sine and cosine terms in the Fourier series of f(x) as given in the following table.*

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Solution.

x	$\frac{x\pi}{3}$	$\sin\frac{\pi x}{3}$	$\cos\frac{\pi x}{3}$	$f(x)$	$f(x)\sin\frac{\pi x}{3}$	$f(x)\cos\frac{\pi x}{3}$
0	0	0	1.0	9	0	9
1	$\frac{\pi}{3}$	0.867	0.5	18	15.606	9
2	$\frac{2\pi}{3}$	0.867	– 0.5	24	20.808	– 12
3	$\frac{3\pi}{3}$	0	– 1.0	28	0	– 28
4	$\frac{4\pi}{3}$	– 0.867	– 0.5	26	– 22.542	– 13
5	$\frac{5\pi}{3}$	– 0.867	0.5	20	– 17.340	10
			$\Sigma =$	125	– 3.468	– 25

$$a_0 = 2 \text{ Mean of } f(x) = 2 \times \frac{125}{6} = 41.66$$

$$a_1 = 2 \text{ Mean of } f(x) \cos\frac{\pi x}{3} = 2 \times \frac{-25}{6} = -8.33$$

$$b_1 = 2 \text{ Mean of } f(x) \sin\frac{\pi x}{3} = 2 \times \frac{-3.468}{6} = -1.156$$

Fourier series is $f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + \ldots + b_1 \sin\frac{\pi x}{3} + \ldots$

$$= 20.83 - 8.33 \cos\frac{\pi x}{3} + \ldots - 1.156 \sin\frac{\pi x}{3} + \ldots$$ **Ans.**

Exercise 12.7

1. In a machine the displacement *f (x)* of a given point is given for a certain angle x^o as follows:

x^o	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of $\sin 2x$ in the Fourier series representing the above variations. **Ans.** – 0.072

2. The displacement $f(x)$ of a part of a machine is tabulated with corresponding angular moment 'x' of the crank. Express $f(x)$ as a Fourier series upto third harmonic.

x^o	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	1.80	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Ans. $f(x) = 1.26 + 0.04 \cos x + 0.53 \cos 2x - 0.1 \cos 3x + \ldots$
$- 0.63 \sin x - 0.23 \sin 2x + 0.085 \sin 3x + \ldots$

3. The following values of y give the displacement in cms of a certain machine part of the rotation x of the flywheel. Expand $f(x)$ in the form of a Fourier series.

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
$f(x)$	0	9.2	14.4	17.8	17.3	11.7

Ans. $f(x) = 11.733 - 7.733 \cos 2x - 2.833 \cos 4x + \ldots$
$- 1.566 \sin 2x - 0.116 \sin 4x + \ldots$

4. Analyse harmonically the data given below and express y in Fourier series upto the second harmonic.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

13

Laplace Transformation

13.1 INTRODUCTION

Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

13.2 LAPLACE TRANSFORM

Definition. Let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided the integral exists, is called the **Laplace Transform** of $f(t)$. It is denoted as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

13.3 IMPORTANT FORMULAE

(1) $L(1) = \dfrac{1}{s}$ (2) $L(t^n) = \dfrac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3....$

(3) $L(e^{at}) = \dfrac{1}{s-a}$ $(s > a)$

(4) $L(\cosh at) = \dfrac{s}{s^2 - a^2}$ $(s^2 > a^2)$

(5) $L(\sinh at) = \dfrac{a}{s^2 - a^2}$ $(s^2 > a^2)$

(6) $L \sin(at) = \dfrac{a}{s^2 + a^2}$ $(s > 0)$

(7) $L(\cos at) = \dfrac{s}{s^2 + a^2}$ $(s > 0)$

1. $L(1) = \dfrac{1}{s}$

Proof. $L(1) = \displaystyle\int_0^\infty 1 \,.\, e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$

Hence $L(1) = \dfrac{1}{s}$ **Proved.**

2. $L(t^n) = \dfrac{n!}{s^{n+1}}$, where n and s are positive.

Proof. $L(t^n) = \displaystyle\int_0^\infty e^{-st} t^n\, dt$

Putting $st = x$ or $t = \dfrac{x}{s}$ or $dt = \dfrac{dx}{s}$

Thus, we have, $L(t^n) = \displaystyle\int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$ or $L(t^n) = \dfrac{1}{s^{n+1}}\displaystyle\int_0^\infty e^{-x} \,.\, x^n\, dx$

or $L(t^n) = \dfrac{n!}{s^{n+1}}$ $\left[\begin{array}{l} \overline{|n+1} = \int_0^\infty e^{-x} \,.\, x^n\, dx \\ \text{and } \overline{|n+1} = n! \end{array}\right]$ **Proved**

3. $L(e^{at}) = \dfrac{1}{s-a}$ where $s > a$

Proof. $L(e^{at}) = \displaystyle\int_0^\infty e^{-st} \,.\, e^{at}\, dt = \int_0^\infty e^{-st+at} \,.\, dt$

$$= \int_0^\infty e^{(-s+a)t} \,.\, dt = \int_0^\infty e^{-(s-a)t} \,.\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$ **Proved**

4. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

Proof. $L(\cosh at) = L\left[\dfrac{e^{at} + e^{-at}}{2}\right]$ $\left(\because \cosh at = \dfrac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$ $\left[L(e^{at}) = \dfrac{1}{s-a}\right]$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$ **Proved.**

5. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

Proof. $L(\sinh at) = L\left[\dfrac{1}{2}(e^{at} - e^{-at})\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$ **Proved.**

6. $L(\sin at) = \dfrac{a}{s^2 + a^2}$

Proof. $L(\sin at) = L\left[\dfrac{e^{iat} - e^{-iat}}{2i}\right]$ $\left[\because \sin at = \dfrac{e^{iat} - e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}[\mathrm{L}(e^{iat} - e^{-iat})] = \frac{1}{2i}[\mathrm{L}(e^{iat}) - \mathrm{L}(e^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\frac{s+ia-s+ia}{s^2+a^2}$$

$$= \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$ **Proved**

7. $\mathrm{L}(\cos at) = \dfrac{s}{s^2+a^2}$

Proof. $$\mathrm{L}(\cos at) = \mathrm{L}\left(\frac{e^{iat}+e^{-iat}}{2}\right) \qquad \left[\because \cos at = \frac{e^{iat}+e^{-iat}}{2}\right]$$

$$= \frac{1}{2}[\mathrm{L}(e^{iat}+e^{-iat})] = \frac{1}{2}[\mathrm{L}(e^{iat}) + \mathrm{L}(e^{-iat})]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2}\frac{s+ia+s-ia}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$ **Proved**

Example 1. *Find the Laplace transform of f(t) defined as*

$$f(t) = \frac{t}{k}, \quad \text{when } 0 < t < k$$
$$= 1, \quad \text{when } t > k$$ *(Mangalore 1997)*

Solution.

$$f(t) = \int_0^k \frac{t}{k} e^{-st}\,dt + \int_k^\infty 1\cdot e^{-st}\,dt = \frac{1}{k}\left[\left(t\frac{e^{-st}}{-s}\right)_0^k - \int_0^k \frac{e^{-st}}{-s}\,dt\right] + \left[\frac{e^{-st}}{-s}\right]_k^\infty$$

$$= \frac{1}{k}\left[\frac{k\,e^{-ks}}{-s} - \left(\frac{e^{-st}}{s^2}\right)_0^k\right] + \frac{e^{-ks}}{s} = \frac{1}{k}\left[\frac{k\,e^{-ks}}{-s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-ks}}{s}$$

$$= -\frac{e^{-sk}}{s} - \frac{1}{k}\frac{e^{-ks}}{s^2} + \frac{1}{k}\frac{1}{s^2} + \frac{e^{-ks}}{s} = \frac{1}{ks^2}[-e^{-ks}+1]$$ **Ans.**

Example 2. *From the first principle, find the Laplace transform of (1 + cos 2 t).*

Solution. Laplace transform of $(1 + \cos 2t)$

$$= \int_0^\infty e^{-st}(1+\cos 2t)\,dt = \int_0^\infty e^{-st}\left(1 + \frac{e^{2it}+e^{-2it}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty [2e^{-st} + e^{(-s+2i)t} + e^{(-s-2i)t}]\,dt = \frac{1}{2}\left[\frac{2e^{-st}}{-s} + \frac{e^{(-s+2i)t}}{-s+2i} + \frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$= \frac{1}{2}\left[\left(0+\frac{2}{s}\right) + \frac{1}{-s+2i}(0-1) + \frac{1}{-s-2i}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{2}{s} + \frac{1}{s-2i} + \frac{1}{s+2i}\right] = \frac{1}{2}\left[\frac{2}{s} + \frac{2s}{s^2+4}\right]$$

$$= \frac{1}{s} + \frac{s}{s^2+4} = \frac{2s^2+4}{s(s^2+4)}$$ **Ans.**

13.4 PROPERTIES OF LAPLACE TRANSFORMS

(1) $\mathrm{L}[a f_1(t) + b f_2(t)] = a\,\mathrm{L}[f_1(t)] + b\,\mathrm{L}[f_2(t)]$

Proof. $$\mathrm{L}[a f_1(t) + b f_2(t)] = \int_0^\infty e^{-st}[a f_1(t) + b f_2(t)]$$

$$= a\int_0^\infty e^{-st} f_1(t)\,dt + b\int_0^\infty e^{-st} f_2(t)\,dt$$

$$= a\,\mathrm{L} f_1(t) + b\,\mathrm{L} f_2(t)$$ **Proved**

(2) **First Shifting Theorem.** If $\mathrm{L} f(t) = \mathrm{F}(s)$, then

$$\mathbf{L}[e^{at} f(t)] = \mathbf{F}(s-a)$$

Proof. $$\mathrm{L}[e^{at} f(t)] = \int_0^\infty e^{-st}\,.\,e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-rt} f(t)\,dt \qquad \text{where } r = s-a$$

$$= F(r) = F(s-a)$$

With the help of this property, we can have the following important results :

(1) $\mathrm{L}(e^{at} t^n) = \dfrac{n!}{(s-a)^{n+1}}$ $\qquad \left[\mathrm{L}(t^n) = \dfrac{n!}{s^{n+1}}\right]$

(2) $\mathrm{L}(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$ (3) $\mathrm{L}(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2}$

(4) $\mathrm{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ (5) $\mathrm{L}(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$

Example 3. *Find the Laplace transform of $\cos^2 t$.*

Solution. $$\cos 2t = 2\cos^2 t - 1$$

$\therefore$ $$\cos^2 t = \frac{1}{2}[\cos 2t + 1]$$

$$\mathrm{L}(\cos^2 t) = \mathrm{L}\left[\frac{1}{2}(\cos 2t + 1)\right] = \frac{1}{2}[\mathrm{L}(\cos 2t) + \mathrm{L}(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(2)^2} + \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$$ **Ans.**

Example 4. *Find the Laplace Transform of $t^{-\frac{1}{2}}$.*

Solution. We know that $\mathrm{L}(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -\dfrac{1}{2}$, $\mathrm{L}(t^{-1/2}) = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}} = \dfrac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$ where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **Ans.**

Example 5. *Find the Laplace Trnasform of t sin at.*

Solution.

$$L(t\sin at) = L\left(t\,\frac{e^{iat}-e^{-iat}}{2i}\right) = \frac{1}{2i}[L(t\,.\,e^{iat}) - L(te^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right] = \frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2(s+ia)^2}\right]$$

$$= \frac{1}{2i}\,\frac{(s^2+2\,ias-a^2)-(s^2-2\,ias-a^2)}{(s^2+a^2)^2}$$

$$= \frac{1}{2i}\,\frac{4\,ias}{(s^2+a^2)^2} = \frac{2\,as}{(s^2+a^2)^2} \qquad \textbf{Ans.}$$

Example 6. *Find the Laplace Transform of t^2 cos at.*

Solution.

$$L(t^2\cos at) = L\left(t^2\,.\,\frac{e^{iat}+e^{-iat}}{2}\right) = \frac{1}{2}[L(t^2\,.\,e^{iat}) + L(t^2e^{-iat})]$$

$$= \frac{1}{2}\left[\frac{2\,!}{(s-ia)^3} + \frac{2\,!}{(s+ia)^3}\right] = \frac{(s+ia)^3+(s-ia)^3}{(s-ia)^3(s+ia)^3}$$

$$= \frac{(s^3+3\,ias^2-3\,a^2s-ia^3)+(s^3-3\,ias^2-3\,a^2s+ia^3)}{(s^2+a^2)^3}$$

$$= \frac{2\,s^3-6\,a^2s}{(s^2+a^2)^3} = \frac{2\,s\,(s^2-3\,a^2)}{(s^2+a^2)^3} \qquad \textbf{Ans.}$$

EXERCISE 13.1

Find the Laplace Transforms of the following:

1. $t+t^2+t^3$ **Ans.** $\frac{1}{s^2}+\frac{2}{s^3}+\frac{6}{s^4}$.
2. $\sin t\cos t$ **Ans.** $\frac{1}{s^2+4}$.
3. t^3e^{-2t} **Ans.** $\frac{6}{(s+2)^4}$.
4. $\sin^3 2t$ **Ans.** $\frac{48}{(s^2+4)(s^2+36)}$
5. $e^{-t}\cos^2 t$ **Ans.** $\frac{1}{2s+2}+\frac{s+1}{2s^2+4s+10}$
6. $\sin 2t\cos 3t$ **Ans.** $\frac{2(s^2-5)}{(s^2+1)(s^2+25)}$
7. $\sin 2t\sin 3t$ **Ans.** $\frac{12\,s}{(s^2+1)(s^2+25)}$
8. $\cos at\sinh at$ **Ans.** $\frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}-\frac{s+a}{(s+a)^2+a^2}\right]$
9. $\sinh^3 t$ **Ans.** $\frac{6}{(s^2-1)(s^2-9)}$
10. $\cos t\cos 2t$ **Ans.** $\frac{s(s^2+5)}{(s^2+1)(s^2+9)}$
11. $\cosh at\sin at$ **Ans.** $\frac{a(s^2+2a^2)}{s^4+4a^4}$
12. $f(t) = \begin{cases} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3 \end{cases}$ **Ans.** $\frac{2}{s^3}-\frac{e^{-2s}}{s^3}(2+3s+3s^2)+\frac{e^{-3s}}{s^2}(5s-1)$
13. $f(t) = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right), & t>\frac{2\pi}{3} \\ 0 & t<\frac{2\pi}{3} \end{cases}$ **Ans.** $e^{\frac{-2\pi s}{3}}\cdot\frac{s}{s^2+1}$

13.5 LAPLACE TRANSFORM OF THE DERIVATIVE OF $f(t)$

$$\mathbf{L[f'(t)] = s\,L[f(t)] - f(0)} \qquad \text{where } L[f(t)] = F(s).$$

Proof. $L[f'(t)] = \int_0^\infty e^{-st}f'(t)\,dt$

Integrating by parts, we get

$$L[f'(t)] = \left[e^{-st}\,.\,f(t)\right]_0^\infty - \int_0^\infty (-se^{-st})f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st}f(t)\,dt \qquad (e^{-st}f(t) = 0, \text{ when } t=\infty)$$

$$= -f(0) + s\,Lf(t)$$

or $\quad L[f'(t)] = s\,Lf(t) - f(0).$ **Proved**

Note. Roughly, Laplace transform of **derivative** of $f(t)$ corresponds to **multiplication** of the Laplace transform of $f(t)$ by s.

13.6 LAPLACE TRANSFORM OF DERIVATIVE OF ORDER *n*.

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \ldots - f^{n-1}(0).$$

Proof. We have already proved in Article 13.5 that

$$L[f'(t)] = s\,L[f(t)] - f(0) \qquad \ldots(1)$$

Replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (1) we get

$$L[f''(t)] = s\,L[f'(t)] - f'(0). \qquad \ldots(2)$$

Putting the value of $L[f'(t)]$ from (1) in (2), we have

$$L[f''(t)] = s[s\,L[f(t)] - f(0)] - f'(0)$$

or $\quad L[f''(t)] = s^2L[f(t)] - s\,f(0) - f'(0)$

Similarly $\quad L[f'''(t)] = s^3L[f(t)] - s^2f(0) - s\,f'(0) - f''(0)$

$$L[f^{iv}(t)] = s^4L[f(t)] - s^3f(0) - s^2f'(0) - s\,f''(0) - f'''(0)$$

$$\ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots \qquad \ldots$$

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \ldots - f^{n-1}(0)$$

13.7 LAPLACE TRANSFORM OF INTEGRAL OF $f(t)$

$$\mathbf{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)}, \qquad \text{where } L[f(t)] = F(s).$$

Proof. Let $\quad \phi(t) = \int_0^t f(t)\,dt \;$ and $\; \phi(0) = 0 \;$ then $\; \phi'(t) = f(t)$

We know the formula of Laplace transforms of $\phi'(t)$ *i.e.*

$$L[\phi'(t)] = s\,L[\phi(t)] - \phi(0)$$

or $\quad L[\phi'(t)] = s\,L[\phi(t)] \qquad [\phi(0) = 0]$

or $\quad L[\phi(t)] = \frac{1}{s}L[\phi'(t)]$

Putting the values of $\phi(t)$ and $\phi'(t)$ we get

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}L[f(t)] \quad \text{or} \quad L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s) \qquad \textbf{Proved.}$$

Note. (1) Laplace Transform of **Integral** of $f(t)$ corresponds to the division of the Laplace transform of $f(t)$ by s.

(2) $$\int_0^t f(t) = L^{-1}\left[\frac{1}{s}F(s)\right]$$

13.8 LAPLACE TRANSFORM OF *t. f (t)* (Multiplication by *t*)

If $$L[f(t)] = F(s), \quad \text{then}$$

$$\boldsymbol{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)].}$$

Proof. $$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad ...(1)$$

Differentiating (1) w.r.t. "s" we get

$$\therefore \quad \frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty (-t\,e^{-st}).f(t)\,dt = \int_0^\infty e^{-st}[-t.f(t)]\,dt$$

$$= L[-t f(t)] \quad \text{or} \quad L[t f(t)] = (-1)^1 \frac{d}{ds}[F(s)]$$

Similarly $$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$ **Proved.**

Example 7. *Find the Laplace transform of t sinh at.*

Solution. $$L(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\therefore \quad L[t \sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right)$$

or $$L[t \sinh at] = \frac{2as}{(s^2 - a^2)^2}$$ **Ans.**

Example 8. *Find the Laplace transform of t^2 cos at.*

Solution. $$L(\cos at) = \frac{s}{s^2 + a^2}$$

$$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2 + a^2}\right] = \frac{d}{ds}\frac{(s^2 + a^2).1 - s(2s)}{(s^2 + a^2)^2} = \frac{d}{ds}\frac{a^2 - s^2}{(s^2 + a^2)^2}$$

$$= \frac{(s^2 + a^2)^2(-2s) - (a^2 - s^2).2(s^2 + a^2)(2s)}{(s^2 + a^2)^4} = \frac{-2s^3 - 2a^2 s - 4a^2 s + 4s^3}{(s^2 + a^2)^3}$$

$$= \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3}$$ **Ans.**

Example 9. *Obtain the Laplace transform of*

$t^2 e^t$. sin 4 t.

Solution. $$L(\sin 4t) = \frac{4}{s^2 + 16}, \quad L(e^t \sin 4t) = \frac{4}{(s-1)^2 + 16}$$

$$L(t\,e^t \sin 4t) = -\frac{d}{ds}\frac{4}{s^2 - 2s + 17} = \frac{4(2s - 2)}{(s^2 - 2s + 17)^2}$$

$$L(t^2 e^t \sin 4t) = -4\frac{d}{ds}\frac{2s - 2}{(s^2 - 2s + 17)^2}$$

$$= -4\frac{(s^2 - 2s + 17)^2\,2 - (2s - 2)\,2\,(s^2 - 2s + 17)(2s - 2)}{(s^2 - 2s + 17)^4}$$

$$= \frac{-4(2s^2 - 4s + 34 - 8s^2 + 16s - 8}{(s^2 - 2s + 17)^3}$$

$$= \frac{-4(-6s^2 + 12s + 26)}{(s^2 - 2s + 17)^3} = \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$$ **Ans.**

EXERCISE 13.2

Find the Laplace transforms of the following :

1. $t \sin 2t$ (*Madras 2006*) **Ans.** $\frac{4s}{(s^2 + 4)^2}$ **2.** $t \sin at$ **Ans.** $\frac{2as}{(s^2 + a^2)^2}$ **3.** $t \cosh at$ **Ans.** $\frac{s^2 + a^2}{(s^2 - a^2)^2}$

4. $t \cos t$ **Ans.** $\frac{s^2 - 1}{(s^2 + 1)^2}$ **5.** $t \cosh t$ **Ans.** $\frac{s^2 + 1}{(s^2 - 1)^2}$

6. $t^2 \sin t$ **Ans.** $\frac{2(3s^2 - 1)}{(s^2 + 1)^3}$ **7.** $t^3 e^{-3t}$ **Ans.** $\frac{6}{(s + 3)^4}$

8. $t \sin^2 3t$ **Ans.** $\frac{1}{2}\left[\frac{1}{s^2} - \frac{s^2 - 36}{(s^2 + 36)^2}\right]$ **9.** $t\,e^{at} \sin at$ **Ans.** $\frac{2a(s - a)}{(s^2 - 2as + 2a^2)^2}$

10. $\int_0^t e^{-2t}\, t \sin^3 t\, dt.$ **Ans.** $\frac{3(s + 2)}{2s}\left[\frac{1}{[(s + 2)^2 + 9]^2} - \frac{1}{[(s + 2)^2 + 1]^2}\right]$

11. $t\,e^{-t} \cosh t$ **Ans.** $\frac{s^2 + 2s + 2}{(s^2 + 2s)^2}$

12. $t^2 e^{-2t} \cos t$ **Ans.** $\frac{2(s^3 + 10s^2 + 25s + 22)}{(s^2 + 4s + 5)^3}$

13. (*a*) Laplace transform of $t^n e^{-at}$ is

(*i*) $\frac{\Gamma n}{(s + a)^n}$ (*ii*) $\frac{(n + 1)!}{(s + a)^{n+1}}$ (*iii*) $\frac{n!}{(s + a)^n}$ (*iv*) $\frac{\overline{|n + 1}}{(s + a)^{n+1}}$ **Ans.** (*iv*)

(*b*) Laplace transform of $f(t) = t\,e^{at} \cdot \sin(at),\ t > 0$

(*i*) $\frac{2a(s - a)}{[(s - a)^2 + a^2]^2}$ (*ii*) $\frac{a(s - a)}{(s - a)^2 + a^2}$ (*iii*) $\frac{s - a}{(s - a)^2 + a^2}$ (*iv*) $\frac{(s - a)^2}{(s - a)^2 + a^2}$ **Ans.** (*i*)

(*c*) If $f(x) = x^4 P(x)$, where $P(x)$ has derivatives of all orders, then $L\left[\frac{d^4 f(x)}{dx^4}\right]$ is given by

(*i*) $s^3 L[f(x)]$ (*ii*) $s^4 L f(x)$ (*iii*) $s^4 L[f^3(x)]$ (*iv*) none of these. **Ans.** (*ii*)

(*d*) The Laplace transform of $te^{-t} \cosh 2t$ is

(*i*) $+\frac{s^2 + 2s + 5}{(s^2 + 2s - 3)^2}$; (*ii*) $\frac{s^2 - 2s + 5}{(s^2 + 2s - 3)^2}$;

(*iii*) $\frac{4s + 4}{(s^2 + 2s - 3)^2}$; (*iv*) $\frac{4s - 4}{(s^2 + 2s - 3)^2}$. **Ans.** (*i*)

13.9 LAPLACE TRANSFORM OF $\frac{1}{t} f(t)$ (Division by t)

If $L[f(t)] = F(s)$, then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\, ds$

Proof. $L[f(t)] = F(s)$ or $F(s) = \int_0^\infty e^{-t} f(t)\, dt$ ...(1)

Integrating (1) w.r.t. 's', we have

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt = \int_0^\infty \left[\frac{e^{-st} f(t)}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty \frac{-f(t)}{t}\left[e^{-st}\right]_s^\infty dt = \int_0^\infty \frac{-f(t)}{t}\left[0 - e^{-st}\right] dt$$

$$= \int_0^\infty e^{-st}\left\{\frac{1}{t} \cdot f(t)\right\} dt = L\left[\frac{1}{t} f(t)\right]$$

or $$L\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\, ds.$$ **Proved**

Cor. $L^{-1} \int_s^\infty F(s)\, ds = \frac{1}{t} f(t)$

Example 10. *Find the Laplace transform of* $\frac{\sin 2t}{t}$.

Solution. $L(\sin 2t) = \frac{2}{s^2 + 4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}\, ds = 2 \cdot \frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$ **Ans.**

Example 11. *Find the Laplace transform of* $f(t) = \int_0^t \frac{\sin t}{t}\, dt$.

Solution. $L \sin t = \frac{1}{s^2+1}$

$$L\frac{\sin t}{t} = \int_s^\infty \frac{1}{s^2+1}\, ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\int_0^t \frac{\sin t}{t}\, dt = \frac{1}{s}\cot^{-1} s$$ **Ans.**

Example 12. *Find the Laplace transform of* $\frac{1 - \cos t}{t^2}$

Solution. $L(1 - \cos t) = L(1) - L(\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$

$$L\frac{(1-\cos t)}{t} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2\left(1+\frac{1}{s^2}\right)}\right]_s^\infty = \frac{1}{2}\left[0 - \log\frac{s^2}{s^2+1}\right] = -\frac{1}{2}\log\frac{s^2}{s^2+1}$$

Again, $$L\left[\frac{1-\cos t}{t^2}\right] = -\frac{1}{2}\int_s^\infty \log\frac{s^2}{s^2+1}\, ds = -\frac{1}{2}\int_s^\infty \left(\log\frac{s^2}{s^2+1} \cdot 1\right) ds$$

Integrating by parts, we have

$$= -\frac{1}{2}\left[\log\frac{s^2}{s^2+1} \cdot s - \int \frac{s^2+1}{s^2}\,\frac{(s^2+1)\,2s - s^2(2s)}{(s^2+1)^2} \cdot s\, ds\right]_s^\infty$$

$$= -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\int\frac{1}{s^2+1}\, ds\right]_s^\infty = -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\tan^{-1} s\right]_s^\infty$$

$$= -\frac{1}{2}\left[0 - 2\left(\frac{\pi}{2}\right) - s\log\frac{s^2}{s^2+1} + 2\tan^{-1} s\right] = -\frac{1}{2}\left[-\pi - s\log\frac{s^2}{s^2+1} + 2\tan^{-1} s\right]$$

$$= \frac{\pi}{2} + \frac{s}{2}\log\frac{s^2}{s^2+1} - \tan^{-1} s$$

$$= \left(\frac{\pi}{2} - \tan^{-1} s\right) + \frac{s}{2}\log\frac{s^2}{s^2+1} = \cot^{-1} s + \frac{s}{2}\log\frac{s^2}{s^2+1}.$$ **Ans.**

Example 13. *Evaluate* $L\left[e^{-4t}\frac{\sin 3t}{t}\right]$.

Solution. $$L\sin 3t = \frac{3}{s^2+3^2} \Rightarrow L\frac{\sin 3t}{t} = \int_s^\infty \frac{3}{s^2+9}\, ds = \left[\frac{3}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

$$L\left[e^{-4t}\,\frac{\sin 3t}{t}\right] = \cot^{-1}\frac{s+4}{3} = \tan^{-1}\frac{3}{s+4}$$ **Ans.**

Exercise 13.3

Find the Laplace Transform of the following :

1. $\frac{1}{t}(1 - e^t)$ **Ans.** $\log\frac{s-1}{s}$
2. $\frac{1}{t}(e^{-at} - e^{-bt})$ (*Kuvempu 1996S*) **Ans.** $\log\frac{s+b}{s+a}$
3. $\frac{1}{t}(1 - \cos at)$ (*Mysore 1997S*) **Ans.** $-\frac{1}{2}\log\frac{s^2}{s^2+a^2}$
4. $\frac{1}{t}(\cos at - \cos bt)$ (*Madras 1997*) **Ans.** $-\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$

5. $\frac{1}{t}\sin^2 t$ **Ans.** $\frac{1}{4}\log\frac{s^2+4}{s^2}$ **6.** $\frac{1}{t}\sinh t$ **Ans.** $-\frac{1}{2}\log\frac{s-1}{s+1}$

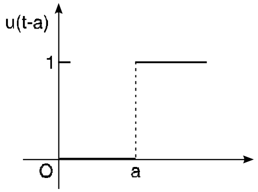
7. $\frac{1}{t}(e^{-t}\sin t)$ **Ans.** $\cot^{-1}(s+1)$

8. $\frac{1}{t}(1-\cos t)$ **Ans.** $\frac{1}{2}\log(s^2+1)-\log s$

9. $\int_0^\infty t\,e^{-2t}\sin t\,dt$ **Ans.** $\frac{4}{25}$ **10.** $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$ **Ans.** $\log 3$

13.10 UNIT STEP FUNCTION

With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.



The unit step functions $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases} \qquad \text{where } a \geq 0.$$

Example 14. *Express the following function in terms of units step functions and find its Laplace transform*:

$$f(t) = \begin{bmatrix} 8, & t < 2 \\ 6, & t > 2 \end{bmatrix}$$

Solution.

$$f(t) = \begin{bmatrix} 8+0 & t < 2 \\ 8-2 & t > 2 \end{bmatrix}$$

$$= 8+\begin{bmatrix} 0 & t < 2 \\ -2 & t > 2 \end{bmatrix} = 8+(-2)\begin{bmatrix} 0, & t < 2 \\ 1, & t > 2 \end{bmatrix}$$

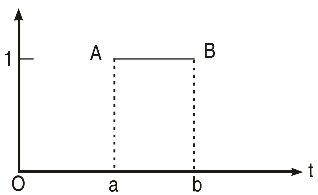
$$= 8-2\,u(t-2)$$

$$\mathrm{L}\,f(t) = 8\,\mathrm{L}(1)-2\,\mathrm{L}\,u(t-2) = \frac{8}{s}-2\frac{e^{-2s}}{s} \qquad \textbf{Ans.}$$

Example 15. *Draw the graph of* $u(t-a)-u(t-b)$

Solution. As in Art 13.10 the graph of $u(t-a)$ is a straight line from A to ∞. Similarly, the graph of $u(t-b)$ a straight line from B to ∞.

Hence, the graph of $u[t-a]-u[t-b]$ is AB.



Example 16. *Express the following function in terms of unit step function and find its Laplace transform :*

$$f(t) = \begin{bmatrix} E & a < t < b \\ 0 & t > b \end{bmatrix}$$

Solution.

$$f(t) = E\begin{bmatrix} 1 & a < t < b \\ 0 & t > b \end{bmatrix} = E\,[u(t-a)-u(t-b)]$$

$$\mathrm{L}\,f(t) = E\left[\frac{e^{-as}}{s}-\frac{e^{-bs}}{s}\right] \qquad \textbf{Ans.}$$

Example 17. *Express the following function in terms of unit step function :*

$$f(t) = \begin{bmatrix} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{bmatrix}$$

and find its Laplace transform.

Solution.

$$f(t) = \begin{bmatrix} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{bmatrix}$$

$$= (t-1)\,[u(t-1)-u(t-2)]+(3-t)\,[u(t-2)-u(t-3)]$$

$$= (t-1)\,u(t-1)-(t-1)\,u(t-2)+(3-t)\,u(t-2)+(t-3)\,u(t-3)$$

$$= (t-1)\,u(t-1)-2\,(t-2)\,u(t-2)+(t-3)\,u(t-3)$$

$$\mathrm{L}\,f(t) = \frac{e^{-s}}{s^2}-2\frac{e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2} \qquad \textbf{Ans.}$$

Laplace Transform of unit function

$$\mathrm{L}\,[u(t-a)] = \frac{e^{-as}}{s}.$$

Proof.

$$\mathrm{L}\,[u(t-a)] = \int_0^\infty e^{-st}\,u(t-a)\,dt$$

$$= \int_0^a e^{-st}.\,0\,dt+\int_a^\infty e^{-st}.1\,dt = 0+\left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$\therefore \quad \mathrm{L}\,[u(t-a)] = \frac{e^{-as}}{s} \qquad \textbf{Proved.}$$

13.11 SECOND SHIFTING THEOREM

If $\mathrm{L}\,[f(t)] = F(s)$, then $\mathbf{L\,[f(t-a)\,.\,u(t-a)} = \mathbf{e^{-as}\,F(s)}$.

Proof. $\mathrm{L}\,[f(t-a)\,.\,u(t-a)] = \int_0^\infty e^{-st}\,[f(t-a)\,.\,u(t-a)]\,dt$

$$= \int_0^a e^{-st}f(t-a)\,.\,0\,dt+\int_0^\infty e^{-st}f(t-a)\,(1)\,dt$$

$$= \int_0^\infty e^{-st}f(t-a)\,dt$$

$$= \int_0^\infty e^{-s(u+a)}f(u)\,du, \qquad \text{where } u = t-a$$

$$= e^{-sa}\int_0^\infty e^{-su}.\,f(u)\,du = e^{-sa}F(s) \qquad \textbf{Proved.}$$

13.12 THEOREM $\mathrm{L}\,f(t)\,u(t-a) = e^{-as}\,\mathrm{L}\,[f(t+a)]$

Proof.

$$\mathrm{L}\,f(t)\,.\,u(t-a) = \int_0^\infty e^{-st}\,[f(t)\,.\,u(t-a)]\,dt$$

$$= \int_0^a e^{-st}\,[f(t)\,.\,u(t-a)]\,dt+\int_a^\infty e^{-st}.\,[f(t)\,.\,u(t-a)]\,dt$$

$$= 0+\int_a^\infty e^{-st}.\,f(t)\,(1)\,dt$$

$$= \int_0^\infty e^{-s(y+a)} . f(y+a)\, dy = e^{-as} \int_0^\infty e^{-sy} . f(y+a)\, dy. \qquad (t-a=y)$$

$$= e^{-as} \int_0^\infty e^{-st} f(t+a) dt = e^{-as} \mathrm{L} f(t+a) \qquad \textbf{Proved}$$

Example 18. *Find the Laplace Transform of $t^2\, u(t-3)$.*

Solution.

$$\begin{aligned} t^2 . u(t-3) &= [(t-3)^2 + 6(t-3) + 9]\, u(t-3) \\ &= (t-3)^2 . u(t-3) + 6(t-3) . u(t-3) + 9\, u(t-3) \\ \mathrm{L}\, t^2 . u(t-3) &= \mathrm{L}\,(t-3)^2 . u(t-3) + 6\,\mathrm{L}\,(t-3) . u(t-3) + 9\,\mathrm{L}\, u(t-3) \\ &= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \qquad \textbf{Ans.} \end{aligned}$$

Aliter

$$\begin{aligned} \mathrm{L}\, t^2\, u(t-3) &= e^{-3s}\,\mathrm{L}\,(t+3)^2 = e^{-3s}\,\mathrm{L}\,[t^2 + 6t + 9] \\ &= e^{3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \qquad \textbf{Ans.} \end{aligned}$$

Example 19. *Find the Laplace transform of $e^{-2t} u_\pi(t)$.*

where
$$u_\pi(t) = \begin{cases} 0; & t < \pi \\ 1; & t > \pi \end{cases}$$

Solution.

$$\begin{aligned} u_\pi(t) &= \begin{cases} 0; & t < \pi \\ 1; & t > \pi \end{cases} \\ &= u(t-\pi) \\ \mathrm{L}\, e^{-2t} u_\pi(t) &= \mathrm{L}\, e^{-2t} u(t-\pi) & f(t) &= e^{-2t} \\ &= e^{-\pi s}\,\mathrm{L}\, f(t+\pi) & f(t+\pi) &= e^{-2(t+\pi)} \\ &= e^{-\pi s}\,\mathrm{L}\, e^{-2(t+\pi)} = e^{-\pi s} e^{-2\pi}\,\mathrm{L}\, e^{-2t} \\ &= e^{-(\pi s + 2\pi)} \frac{1}{s+2} \\ &= \frac{e^{-\pi(s+2)}}{s+2} \qquad \textbf{Ans.} \end{aligned}$$

Example 20. *Represent $f(t) = \sin 2t$, $2\pi < t < 4\pi$* and *$f(t) = 0$ otherwise, in terms of unit step function and then find its Laplace transform.*

Solution.

$$\begin{aligned} f(t) &= \begin{cases} \sin 2t, & 2\pi < t < 4\pi \\ 0, & \text{otherwise} \end{cases} \\ f(t) &= \sin 2t\, [u(t-2\pi) - u(t-4\pi)] \\ \mathrm{L} f(t) &= \mathrm{L}\,[\sin 2t . u(t-2\pi)] - \mathrm{L}\,[\sin 2t . u(t-4\pi)] \\ &= e^{-2\pi s}\,\mathrm{L}\,[\sin 2(t+2\pi)] - e^{-4\pi s}\,\mathrm{L}\,[\sin 2(t+4\pi)] \\ &= e^{-2\pi s}\,\mathrm{L}\,[\sin 2t] - e^{-4\pi s}\,\mathrm{L}\,[\sin(2t)] \\ &= e^{-2\pi s}\frac{2}{s^2+4} - e^{-4\pi s}\frac{2}{s^2+4} \\ &= (e^{-2\pi s} - e^{-4\pi s})\frac{2}{s^2+4} \qquad \textbf{Ans.} \end{aligned}$$

Exercise 13.4

Find the Laplace transform of the following:

1. $f(t) = \begin{cases} t-1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$ **Ans.** $\dfrac{e^{-s} - e^{-2s}}{s^2} - \dfrac{e^{-2s}}{s}$ **2.** $e^t\, u(t-1)$ **Ans.** $\dfrac{e^{-(s-1)}}{s-1}$

3. $t\, u_2(t)$ (*A.M.I.E.T.E., Winter 1996*) **Ans.** $\left(\dfrac{1}{s^2} + \dfrac{1}{s}\right) e^{-2s}$

4. $\dfrac{1 - e^{2t}}{5} + tu(t) + \cosh t \cdot \cos t$ **Ans.** $\log \dfrac{s-2}{s} + \dfrac{1}{s^2} + \dfrac{s^3}{s^4+4}$ **5.** $t^2\, u(t-2)$ **Ans.** $\dfrac{e^{-2s}}{s^3}(4s^2 + 4s + 2)$

6. $\sin t\, u(t-4)$ **Ans.** $\dfrac{e^{-4s}}{s^2+1}[\cos 4 + s \sin 4]$

7. $f(t) = K(t-2)\,[U(t-2) - U(t-3)]$ **Ans.** $\dfrac{K}{s^2}\{e^{-2s} - (s+1)\, e^{-3s}\}$

8. $f(t) = K \dfrac{\sin \pi t}{T}[U(t-2T) - U(t-3T)]$ **Ans.** $\dfrac{K\pi T}{s^2 T^2 + \pi^2}\left(e^{-2sT} - e^{-3sT}\right)$

Express the following in terms of unit step functions and obtain Laplace transforms.

9. $f(t) = \begin{cases} t & 0 < t < 2 \\ 0 & \end{cases}$ **Ans.** $U(t) - U(t-2)$, $\dfrac{1 - (2s+1)\, e^{-2s}}{s^2}$

10. $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ t & t > \pi \end{cases}$ **Ans.** $\dfrac{1 + e^{-\pi s}}{s^2+1} + \dfrac{e^{-\pi s}(\pi s + 1)}{s^2}$

11. $f(t) = \begin{cases} 4 & 0 < t < 1 \\ -2 & 1 < t < 3 \\ 5 & t > 3 \end{cases}$ **Ans.** $\dfrac{4 - 6\, e^{-s} + 7\, e^{-3s}}{s}$

12. The Laplace transform of $t\, u_2(t)$ is

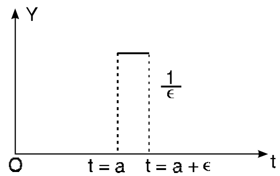
(*i*) $\left(\dfrac{1}{s^2} + \dfrac{2}{s}\right) e^{-2s}$ (*ii*) $\dfrac{1}{s^2} e^{-2s}$ (*iii*) $\left(\dfrac{1}{s^2} - \dfrac{2}{s}\right) e^{-2s}$ (*iv*) $\dfrac{e^{-2s}}{s^2}$

(*A.M.I.E.T.E., Winter 1996*) **Ans.** (*i*)

13.13 (1) IMPULSE FUNCTION

When a large force acts for a short time, then the product of the force and the time is called impulse in applied mechanics. The unit impulse function is the limiting function.

$$\begin{aligned} \delta(t-1) &= \frac{1}{\varepsilon}, a < t < a + \varepsilon \\ &= 0, \quad \text{otherwise} \end{aligned}$$



The value of the function (height of the strip in the figure) becomes infinite as $\varepsilon \to 0$ and the area of the rectangle is unity.

(2) The Unit Impulse function is defined as follows:

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a. \end{cases}$$

and
$$\int_0^\infty \delta(t-a) . dt = 1. \qquad [\text{Area of strip} = 1]$$

(3) Laplace Transform of unit Impulse function

$$\int_0^\infty f(t)\, \delta(t-a)\, dt = \int_a^{a+\varepsilon} f(t) . \frac{1}{\varepsilon} dt \qquad \left\{ \begin{array}{l} \text{Mean value Theorem} \\ \displaystyle\int_a^b f(t)\, dt = (b-a) f(\eta) \end{array} \right.$$

$$= (a+\varepsilon-a)\,f(\eta).\frac{1}{\varepsilon} \qquad \text{where } a < \eta < a+\varepsilon$$

$$= f(\eta)$$

Property I: $\int_0^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$ as $\varepsilon \to 0$

Note. If $f(t) = e^{-st}$ and $L[\delta(t-a)] = e^{-as}$

Example 21. *Evaluate* $\int_{-\infty}^{\infty} e^{-5t}\,\delta(t-2)$.

Solution. $\int_{-\infty}^{\infty} e^{-5t}\,\delta(t-2)\;\; e^{-5\times 2} = e^{-10}$

Property II: $\int_{-\infty}^{\infty} f(t)\,\delta'(t-a)\,dt = -f'(a)$

Proof. $\int_{-\infty}^{\infty} f(t)\,\delta'(t-a)\,dt = [f(t)\cdot\delta(t-a)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t-a)\,dt$

$$= 0 - 0 - f'(a) = -f'(a)$$

Example 22. *Find the Laplace transform of* $t^3\,\delta(t-4)$.

Solution. $L\, t^3\,\delta(t-4) = \int_0^{\infty} e^{-st}\, t^3\,\delta(t-4)\,dt$

$$= 4^3\, e^{-4s}$$ **Ans.**

Exercise 13.5

Evaluate the following:

1. $\int_0^{\infty} e^{-3t}\,\delta(t-4)\,dt$ **Ans.** e^{-12}
2. $\int_{-\infty}^{\infty} \sin 2t\,\delta\left(t-\frac{\pi}{4}\right)$ **Ans.** 1
3. $\int_{-\infty}^{\infty} e^{-3t}\,\delta'(t-2)$ **Ans.** $3\,e^{-6}$
4. $\frac{\delta(t-4)}{t}$ **Ans.** $\frac{e^{-4s}}{4}$
5. Laplace transforms of $\cos t \log t\,\delta(t-\pi)$ **Ans.** $-e^{-\pi s}\log\pi$
6. $e^{-4t}\,\delta(t-3)$ **Ans.** $e^{-3(s+4)}$.

13.14 PERIODIC FUNCTIONS

Let $f(t)$ be a periodic function with Period T, then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Proof. $L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \ldots$$

Substituting $t = u+T$ in second integral and $t = u+2T$ in third integral, and so on.

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \ldots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \ldots$$

$$[f(u) = f(u+T) = f(u+2T) = f(u+3T) = \ldots]$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \ldots$$

$$= [1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots]\int_0^T e^{-st} f(t)\,dt \qquad \left[1 + a + a^2 + a^3 + \ldots = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$ **Proved.**

Example 23. *Find the Laplace transform of the waveform*

$$f(t) = \left(\frac{2t}{3}\right),\quad 0 \le t \le 3.$$

Solution. $L[f(t)] = \frac{1}{1-e^{-st}}\int_0^T e^{-st} f(t)\,dt$

$$L\left[\frac{2t}{3}\right] = \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt = \frac{1}{1-e^{-3s}}\frac{2}{3}\left[\frac{t\,e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3\,e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] = \frac{2}{3}\cdot\frac{1}{1-e^{-3s}}\left[\frac{3\,e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2\,e^{-3s}}{-s(1-e^{-3s})} + \frac{2}{3s^2}.$$ **Ans.**

Example 24. *Find the Laplace transform of the function (Half wave rectifier)*

$$f(t) = \begin{cases} \sin\omega t & \text{for } 0 < t < \dfrac{\pi}{\omega} \\ 0 & \text{for } \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega}. \end{cases}$$

Solution. $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt \qquad \left[\begin{array}{l} f(t) \text{ is a periodic function.} \\ T = \dfrac{2\pi}{\omega} \end{array}\right.$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\times 0\times dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt$$

$$\int e^{ax}\sin bx\,dx = e^{ax}\frac{(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{\omega\, e^{-\frac{\pi}{\omega}s}+\omega}{s^2+\omega^2}\right] = \frac{\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{(s^2+\omega^2)\left[1-e^{-\frac{2\pi s}{\omega}}\right]}$$

$$= \frac{\omega}{(s^2+\omega^2)\left[1-e^{-\frac{\pi s}{\omega}}\right]}$$ **Ans.**

Example 25. *Find the Laplace Transform of the Periodic function (saw tooth wave)*

$$f(t) = \frac{kt}{T} \text{ for } 0 < t < T,\ f(t+T) = f(t).$$

Solution. $$\mathrm{L}\,[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\frac{kt}{T}\,dt$$

$$= \frac{1}{1-e^{-sT}}\frac{k}{T}\int_0^T e^{-st}\,.\,t\,dt = \frac{k}{T(1-e^{-sT})}\left[t\,\frac{e^{-st}}{-s}-\int 1\,.\,\frac{e^{-st}}{-s}\,dt\right]_0^T$$ Integrating by parts

$$= \frac{k}{T(1-e^{-sT})}\left[\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^T = \frac{k}{T(1-e^{-sT})}\left[\frac{T\,e^{-sT}}{-s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

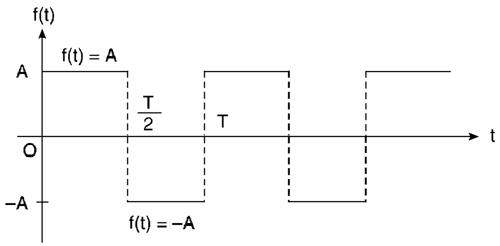
$$= \frac{k}{T(1-e^{-sT})}\left[\frac{T\,e^{-sT}}{-s}+\frac{1}{s^2}(1-e^{-sT})\right]$$

$$= -\frac{k\,e^{-sT}}{s(1-e^{-sT})}+\frac{k}{T s^2}$$ **Ans.**

Example 26. *Obtain Laplace transform of rectangular wave given by*

Solution. $$\mathrm{L} f(t) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$= \frac{\int_0^{\frac{T}{2}} e^{-st} A\,dt+\int_{\frac{T}{2}}^{T} e^{-st}(-A)\,dt}{1-e^{-sT}}$$



$$= A\,\frac{\left[\frac{e^{-st}}{-s}\right]_0^{\frac{T}{2}}-\left[\frac{e^{-st}}{-s}\right]_{\frac{T}{2}}^{T}}{1-e^{-sT}}$$

$$= \frac{A}{1-e^{-sT}}\left[-\frac{e^{-\frac{sT}{2}}}{s}+\frac{1}{s}+\frac{e^{-sT}}{s}-\frac{e^{-\frac{sT}{2}}}{s}\right]$$

$$= \frac{A}{s(1-e^{-sT})}\left[1-2\,e^{-\frac{sT}{2}}+e^{-sT}\right] = \frac{A}{s(1-e^{-sT})}\left[1-e^{-\frac{sT}{2}}\right]^2$$

$$= \frac{A\left[1-e^{-\frac{sT}{2}}\right]^2}{s\left(1+e^{-\frac{sT}{2}}\right)\left(1-e^{-\frac{sT}{2}}\right)} = \frac{A}{s}\,\frac{\left(1-e^{-\frac{sT}{2}}\right)}{\left(1+e^{-\frac{sT}{2}}\right)}$$

$$= \frac{A}{s}\,\frac{\left(e^{\frac{sT}{4}}-e^{-\frac{sT}{4}}\right)}{\left(e^{\frac{sT}{4}}+e^{-\frac{sT}{4}}\right)} = \frac{A}{s}\tanh\frac{sT}{4}$$ **Ans.**

Example 27. *A periodic square wave function $f(t)$, in terms of unit step functions, is written as*

$$f(t) = k\,[u_0(t)-2\,u_a(t)+2\,u_{2a}(t)-2\,u_{3a}(t)+\ldots]$$

Show that the Laplace transform of $f(t)$ is given by

$$L\,[f(t)] = \frac{k}{s}\tanh\left(\frac{as}{2}\right).$$

Solution. $$f(t) = k\,[u_0(t)-2\,u_a(t)+2\,u_{2a}(t)-2\,u_{3a}(t)+\ldots]$$

$$L f(t) = k\,[L\,u_0(t)-2\,L\,u_a(t)+2\,L\,u_{2a}(t)-2\,L\,u_{3a}(t)+\ldots]$$

$$= k\left[\frac{1}{s}-2\,\frac{e^{-as}}{s}+2\,\frac{e^{-2as}}{s}-2\,\frac{e^{-3as}}{s}+\ldots\right]$$

$$= \frac{k}{s}\left[1-2\,e^{-as}+2\,e^{-2as}-2\,e^{-3as}+\ldots\right]$$

$$= \frac{k}{s}\left[1-2\,(e^{-as}-e^{-2as}+e^{-3as}-\ldots)\right]$$

$$= \frac{k}{s}\left[1-2\,\frac{e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{1+e^{-as}-2\,e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}\right] = \frac{k}{s}\tanh\frac{as}{2}$$ **Ans.**

EXERCISE 13.6

1. Find the Laplace Transform of the periodic function

$f(t) = e^t$ for $0 < t < 2\pi$. **Ans.** $\dfrac{e^{2(1-s)\pi}-1}{(1-s)(1-e^{-2\pi s})}$

2. Obtain Laplace transform of full wave rectified sine wave given by

$f(t) = \sin\omega t \qquad 0 < t < \dfrac{\pi}{\omega}$ **Ans.** $\dfrac{\omega}{(s^2+\omega^2)}\coth\dfrac{\pi s}{2\,\omega}$

3. Find the Laplace transform of the staircase function

$f(t) = kn,\ \ np < t < (n+1)\,p,\ n = 0, 1, 2, 3,$ **Ans.** $\dfrac{ke^{ps}}{s(1-e^{-ps})}$

Find Laplace transform of the following:

4. $f(t) = t^2,\ 0 < t < 2,\ f(t+2) = f(t)$ **Ans.** $\dfrac{2-e^{-2s}-4s\,e^{-2s}-4\,s^2\,e^{-2s}}{s^3(1-e^{-2s})}$

5. $f(t) = t, \quad 0 < t < c$

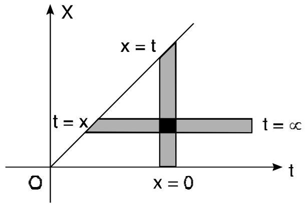
$\quad = 2c-t, \quad c < t < 2c.$ **Ans.** $\dfrac{1}{s^2}\tanh\dfrac{cs}{2}$

6. $f(t) = \begin{cases}\cos\omega t & \text{for } 0 < t < \dfrac{\pi}{\omega}\\ 0 & \text{for } \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega}\end{cases}$ **Ans.** $\dfrac{s}{(s^2+\omega^2)\left(1-e^{\frac{-\pi s}{\omega}}\right)}$

7. $f(t) = \begin{cases} t & , 0 < t < 1 \\ 0 & , 1 < t < 2 \end{cases} \quad f(t+2) = f(t)$ **Ans.** $\dfrac{1 - e^{-s}(s+1)}{s^2(1 - e^{-2s})}$

8. $f(t) = \begin{cases} \dfrac{2t}{T} & , \quad 0 \le t \le \dfrac{T}{2} \\ \dfrac{2}{T}(T - t) , & \dfrac{T}{2} \le t \le T \end{cases} \quad f(t+T) = f(t)$ **Ans.** $\dfrac{2}{Ts^2} \tanh \dfrac{sT}{4} - \dfrac{1}{s\left(e^{\frac{sT}{2}} + 1\right)}$

13.15 CONVOLUTION THEOREM



If $L[f_1(t)] = F_1(s)$ and $L[f_2(t)] = F_2(s)$

then $$L\left\{\int_0^t f_1(x) f_2(t-x)\, dx\right\} = F_1(s) \,.\, F_2(s)$$

or

$$L^{-1} F_1(s) \,.\, F_2(s) = \int_0^t f_1(x) f_2(t-x)\, dx$$

Proof. We have

$$L\left\{\int_0^\infty f_1(x) f_2(t-x)\, dx\right\} = \int_0^\infty e^{-st} \int_0^t f_1(x) f_2(t-x)\, dx\, dt$$

$$= \int_0^\infty \int_0^t e^{-st} f_1(x) f_2(t-x)\, dx\, dt$$

where the double integral is taken over the infinite region in the first quadrant lying between the lines $x = 0$ and $x = t$..

On changing the order of integration, the above integral becomes

$$\int_0^\infty \int_x^\infty e^{-st} f_1(x) f_2(t-x)\, dt\, dx$$

$$= \int_0^\infty e^{-sx} f_1(x)\, dx \int_x^\infty e^{-s(t-x)} f_2(t-x)\, dt$$

$$= \int_0^\infty e^{-sx} f_1(x)\, dx \int_0^\infty e^{-sz} f_2(z)\, dz, \text{ on putting } t - x = z$$

$$= \int_0^\infty e^{-sx} f_1(x)\, F_2(s)\, dx = \left[\int_0^\infty e^{-sx} f_1(x)\, dx\right] F_2(s)$$

$$= F_1(s)\, F_2(s)$$ **Proved.**

13.16 LAPLACE TRANSFORM OF BESSEL FUNCTIONS $J_0(x)$ and $J_1(x)$

Solution. We know that

$$J_0(t) = \left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \,.\, 4^2} - \frac{t^6}{2^2 \,.\, 4^2 \,.\, 6^2} + \ldots\right]$$

Taking Laplace transforms of both sides, we have

$$L\, J_0(t) = \frac{1}{s} - \frac{1}{2^2} \,.\, \frac{2\,!}{s^3} + \frac{1}{2^2 \,.\, 4^2} \frac{4\,!}{s^5} - \frac{1}{2^2 \,.\, 4^2 \,.\, 6^2} \frac{6\,!}{s^7} + \ldots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \ldots\right]$$

$$= \frac{1}{s}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{s^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{1}{s^2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{1}{s^2}\right)^3 + \ldots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-\frac{1}{2}}$$ (By Binomial theorem)

$$= \frac{1}{\sqrt{s^2+1}}$$ **Ans.**

We know that $$L\, f(at) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$\therefore$ $$L\, J_0(at) = \frac{1}{a} \frac{1}{\sqrt{\dfrac{s^2}{a^2} + 1}} = \frac{1}{\sqrt{s^2 + a^2}}$$

$$L\, J_1(x) = -L\, J_0'(x) = -[s\, L\, J_0(x) - J_0(0)]$$

$$= -\left[s \,.\, \frac{1}{\sqrt{s^2+1}} - 1\right] = 1 - \frac{s}{\sqrt{s^2+1}}$$ **Ans.**

13.17 EVALUATION OF INTEGRALS

We can evaluate number of integrals having lower limit 0 and upper limit ∞ by the help of Laplace transform.

Example 28. *Evaluate* $\int_0^\infty t\, e^{-3t} \sin t\, dt$.

Solution. $$\int_0^\infty t\, e^{-3t} \sin t\, dt = \int_0^\infty t\, e^{-st} \sin\, t\, dt \qquad (s = 3)$$

$$= \text{L}\,(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$= \frac{2 \times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$$ **Ans.**

Example 29. *Evaluate* $\int_0^\infty \frac{e^{-t} \sin t}{t} dt$ *and* $\int_0^\infty \frac{\sin t}{t} dt$.

Solution. $$\int_0^\infty \frac{e^{-t} \sin t}{t} dt = \int_0^\infty e^{-st} \frac{\sin t}{t} dt \qquad (s = 1)$$

$$= \text{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} ds = \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s \quad \ldots(1) \qquad = \frac{\pi}{2} - \tan^{-1}(1) \qquad (s = 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$ **Ans.**

On putting $s = 0$ in (1), we get

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1}(0)$$

$$= \frac{\pi}{2}$$ **Ans.**

Exercise 13.7

Evaluate the following by using Laplace Transform.

1. $\int_0^\infty t\, e^{-4t} \sin t\, dt$ **Ans.** $\frac{8}{289}$

2. $\int_0^\infty \frac{e^{-2t} \sinh t \sin t}{t}\, dt$ **Ans.** $\frac{1}{2} \tan^{-1} \frac{1}{2}$

3. $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$ **Ans.** $i\,\frac{5}{2}$

4. $\int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt$ **Ans.** $\log 4$

13.18 FORMULAE OF LAPLACE TRANSFORM

S.No.	$f(t)$	$F(s)$
1.	e^{at}	$\frac{1}{s-a}$
2.	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$
3.	$\sin at$	$\frac{a}{s^2+a^2}$
4.	$\cos at$	$\frac{s}{s^2+a^2}$
5.	$\sinh at$	$\frac{a}{s^2-a^2}$
6.	$\cosh at$	$\frac{s}{s^2-a^2}$
7.	$U(t-a)$	$\frac{e^{-as}}{s}$
8.	$\delta(t-a)$	e^{-as}
9.	$e^{bt} \sin at$	$\frac{a}{(s-b)^2+a^2}$
10.	$e^{bt} \cos at$	$\frac{s-b}{(s-b)^2+a^2}$
11.	$\frac{t}{2a} \sin at$	$\frac{s}{(s^2+a^2)^2}$
12.	$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
13.	$\frac{1}{2a^3}(\sin at - at \cos at)$	$\frac{1}{(s^2+a^2)^2}$
14.	$\frac{1}{2a}(\sin at + at \cos at)$	$\frac{s^2}{(s^2+a^2)^2}$

13.19 PROPERTIES OF LAPLACE TRANSFORM

S.No.	Property	$f(t)$	$F(s)$
1.	Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$ a > 0
2.	Derivative	$\frac{df(t)}{dt}$	$s F(s) - f(0)$ $s > 0$
		$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$ $s > 0$
		$\frac{d^3 f(t)}{dt^3}$	$s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$
3.	Integral	$\int_0^t f(t)\, dt$	$\frac{1}{s} F(s)$ $s > 0$
4.	Initial Value	$\lim_{t \to 0} f(t)$	$\lim_{s \to \infty} sF(s)$
5.	Final Value	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} s F(s)$
6.	First shifting	$e^{-at} f(t)$	$F(s+a)$
7.	Second shifting	$f(t)\, U(t-a)$	$e^{-as}\, \mathrm{L}\, f(t+a)$
8.	Multiplication by t	$t f(t)$	$-\frac{d}{ds} F(s)$
		$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$
9.	Division by t	$\frac{1}{t} f(t)$	$\int_s^\infty F(s)\, ds$
10.	Periodic function	$f(t)$	$\frac{\int_0^T e^{-st} f(t)}{1 - e^{-st}}$üRçQDÇÔçÿ) $f(t+T) = f(t)$
11.	Convolution	$f(t) * g(t)$	$F(s)\, G(s)$

13.20 INVERSE LAPLACE TRANSFORMS

Now we obtain $f(t)$ when $F(s)$ is given, then we say that inverse Laplace transform of $F(s)$ is $f(t)$.

If $\mathrm{L}[f(t)] = F(s)$, then $\mathrm{L}^{-1}[F(s)] = f(t)$.

where L^{-1} is called the inverse Laplace transform operator.

From the application point of view, the inverse Laplace transform is very useful.

13.21 IMPORTANT FORMULAE

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

(2) $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$

(3) $L^{-1}\frac{1}{s-a} = e^{at}$

(4) $L^{-1}\frac{s}{s^2-a^2} = \cosh at$

(5) $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$

(6) $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

(7) $L^{-1}\frac{s}{s^2+a^2} = \cos at$

(8) $L^{-1} F(s-a) = e^{at} f(t)$

(9) $L^{-1}\frac{1}{(s-a)^2+b^2} = \frac{1}{b}e^{at}\sin bt$

(10) $L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$

(11) $L^{-1}\frac{1}{(s-a)^2-b^2} = \frac{1}{b}e^{at}\sinh bt$

(12) $L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$

(13) $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$

(14) $L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$

(15) $L^{-1}\frac{s^2-a^2}{(s^2+a^2)^2} = t\cos at$

(16) $L^{-1}(1) = \delta(t)$

(17) $L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{1}{2a}[\sin at + at\cos at]$

Example 30. *Find the inverse Laplace Transform of the following:*

(*i*) $\frac{1}{s-2}$ (*ii*) $\frac{1}{s^2-9}$ (*iii*) $\frac{s}{s^2-16}$ (*iv*) $\frac{1}{s^2+25}$ (*v*) $\frac{s}{s^2+9}$

(*vi*) $\frac{1}{(s-2)^2+1}$ (*vii*) $\frac{s-1}{(s-1)^2+4}$ (*viii*) $\frac{1}{(s+3)^2-4}$ (*ix*) $\frac{s+2}{(s+2)^2-25}$ (*x*) $\frac{1}{2s-7}$

Solution. (*i*) $L^{-1}\frac{1}{s-2} = e^{2t}$ (*ii*) $L^{-1}\frac{1}{s^2-9} = L^{-1}\frac{1}{3}\cdot\frac{3}{s^2-(3)^2} = \frac{1}{3}\sinh 3t$

(*iii*) $L^{-1}\frac{s}{s^2-16} = L^{-1}\frac{s}{s^2-(4)^2} = \cosh 4t$ (*iv*) $L^{-1}\frac{1}{s^2+25} = \frac{1}{5}\frac{5}{s^2+(5)^2} = \frac{1}{5}\sin 5t$

(*v*) $L^{-1}\frac{s}{s^2+9} = \frac{s}{s^2+(3)^2} = \cos 3t$ (*vi*) $L^{-1}\frac{1}{(s-2)^2+1} = e^{2t}\sin t$

(*vii*) $L^{-1}\frac{s-1}{(s-1)^2+4} = e^{t}\cos 2t$ (*viii*) $L^{-1}\frac{1}{(s+3)^2-4} = \frac{1}{2}\frac{2}{(s+3)^2-(2)^2} = \frac{1}{2}e^{-3t}\sinh 2t$

(*ix*) $L^{-1}\frac{s+2}{(s+2)^2-25} = L^{-1}\frac{(s+2)}{(s+2)^2-(5)^2} = e^{-2t}\cosh 5t$

(*x*) $\frac{1}{2s-7} = \frac{1}{2}e^{\frac{7}{2}t}$ $\left[L^{-1}F(as) = \frac{1}{a}f\left(\frac{t}{a}\right)\right]$

Example 31. *Find the inverse Laplace transform of*

(*i*) $\frac{s^2+s+2}{s^{3/2}}$ (*ii*) $\frac{2s-5}{9s^2-25}$ (*iii*) $\frac{s-2}{6s^2+20}$

Solution.

(*i*) $L^{-1}\frac{s^2+s+2}{s^{3/2}} = L^{-1}s^{1/2} + L^{-1}s^{-1/2} + L^{-1}\frac{2}{s^{3/2}}$

$$= L^{-1}\frac{1}{s^{-1/2}} + L^{-1}\frac{1}{s^{1/2}} + L^{-1}\frac{2}{s^{3/2}} = \frac{t^{-1/2-1}}{\Gamma -\frac{1}{2}} + \frac{t^{1/2-1}}{\Gamma\frac{1}{2}} + \frac{2t^{3/2-1}}{\Gamma\frac{3}{2}}$$

$$= \frac{1}{\Gamma -\frac{1}{2}\; t^{3/2}} + \frac{1}{\sqrt{\pi}\,t} + \frac{4\sqrt{t}}{\sqrt{\pi}}$$ **Ans.**

(*ii*) $$L^{-1}\frac{2s-5}{9s^2-25} = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]} - \frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$

$$= \frac{2}{9}\cosh\frac{5}{3}t - \frac{1}{3}L^{-1}\left(\frac{\frac{5}{3}}{s^2-\left(\frac{5}{3}\right)^2}\right) = \frac{2}{9}\cosh\frac{5t}{3} - \frac{1}{3}\sin\frac{5t}{3}$$ **Ans.**

(*iii*) $$L^{-1}\frac{s-2}{6s^2+20} = L^{-1}\frac{s}{6s^2+20} - L^{-1}\frac{2}{6s^2+20} = \frac{1}{6}L^{-1}\frac{s}{s^2+\frac{10}{3}} - \frac{1}{3}L^{-1}\frac{1}{s^2+\frac{10}{3}}$$

$$= \frac{1}{6}\cos\sqrt{\frac{10}{3}}\,t - \frac{1}{3}\times\sqrt{\frac{3}{10}}\,L^{-1}\frac{\sqrt{\frac{10}{3}}}{s^2+\frac{10}{3}} = \frac{1}{6}\cos\sqrt{\frac{10}{3}}\,t - \frac{1}{\sqrt{30}}\sin\sqrt{\frac{10}{3}}\,t$$ **Ans.**

Exercise 13.8

Find the inverse Laplace transform of the following:

1. $\frac{3s-8}{4s^2+25}$ **Ans.** $\frac{3}{4}\cos\frac{5t}{2} - \frac{4}{5}\sin\frac{5t}{2}$
2. $\frac{3(s^2-2)^2}{2s^5}$ **Ans.** $\frac{3}{2} - 3t^2 + \frac{1}{2}t^4$
3. $\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}$ **Ans.** $\frac{1}{2}\left(\cos\frac{5t}{2} - \sin\frac{5t}{2}\right) - 4\cosh 3t + 6\sinh 3t$
4. $\frac{5s-10}{9s^2-16}$ **Ans.** $\frac{5}{9}\cosh\frac{4}{3}t - \frac{5}{6}\sinh\frac{4}{3}t$
5. $\frac{1}{4s} + \frac{16}{1-s^2}$ Ans. $\frac{1}{4} - 16\sinh t$

13.22 MULTIPLICATION by *s*

$$L^{-1}[s\,F(s)] = \frac{d}{dt}f(t) + f(0)\,\delta(t)$$

Example 32. *Find the inverse Laplace transform of*

(*i*) $\frac{s}{s^2+1}$ (*ii*) $\frac{s}{4s^2-25}$ (*iii*) $\frac{3s}{2s+9}$

Solution. (*i*) $$L^{-1}\frac{1}{s^2+1} = \sin t$$

$$L^{-1}\frac{s}{s^2+1} = \frac{d}{dt}(\sin t) + \sin(0)\,\delta(t)$$

$$= \cos t$$ **Ans.**

(*ii*) $$L^{-1}\frac{1}{4s^2-25} = \frac{1}{4}L^{-1}\frac{1}{s^2-\frac{25}{4}} = \frac{1}{4}\cdot\frac{2}{5}L^{-1}\frac{\frac{5}{2}}{s^2-\left(\frac{5}{2}\right)^2} = \frac{1}{10}\sinh\frac{5}{2}t$$

$$L^{-1}\frac{s}{4s^2-25} = \frac{1}{10}\frac{d}{dt}\sinh\frac{5}{2}t+\frac{1}{10}\sinh\frac{5}{2}(0)$$

$$= \frac{1}{10}\left(\frac{5}{2}\right)\cosh\frac{5}{2}t = \frac{1}{4}\cosh\frac{5}{2}t \qquad \textbf{Ans.}$$

(*iii*) $$L^{-1}\frac{3}{2s+9} = \frac{3}{2}L^{-1}\frac{1}{s+\frac{9}{2}} = \frac{3}{2}e^{-9/2\,t}$$

$$L^{-1}\frac{3s}{2s+9} = \frac{3}{2}\frac{d}{dt}\left(e^{-9/2\,t}\right)+\frac{3}{2}e^{-9/2\,(0)} = \frac{3}{2}\left(-\frac{9}{2}\right)e^{-\frac{11}{2}t}+\frac{3}{2}$$

$$= -\frac{27}{4}e^{-11/2\,t}+\frac{3}{2} \qquad \textbf{Ans.}$$

Exercise 13.9

Find the inverse Laplace transform of the following:

1. $\frac{s}{s+5}$ **Ans.** $-5\,e^{-5t}$ 2. $\frac{2s}{3s+6}$ **Ans.** $-\frac{4}{3}e^{-2t}$

3. $\frac{s}{2s^2-1}$ **Ans.** $\frac{1}{2}\cosh\frac{t}{2}$ 4. $\frac{s^2}{s^2+a^2}$ **Ans.** $-a\sin at+1$

5. $\frac{s^2+4}{s^2+9}$ **Ans.** $-\frac{5}{3}\sin 3t+1$ 6. $\frac{1}{(s-3)^2}$ (*Madras, 2006*) **Ans.** $e^{3t}\,.\,t$

6. $L^{-1}\frac{s^2}{(s^2+4)^2}$ is

(*i*) $\sin 2t+\frac{t}{2}\cos 2t$ (*ii*) $\frac{1}{4}\sin 2t+\frac{t}{2}\cos 2t$ (*iii*) $\frac{1}{4}\sin 2t+t\cos 2t$ (*iv*) $\frac{1}{4}\sin 2t+\frac{t}{4}\cos 2t$

Ans. (*ii*)

13.23 Division by *s* (multiplication by $\frac{1}{s}$)

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt = \int_0^t f(t)\,dt$$

Example 33. *Find the inverse Laplace transform of*

(*i*) $\frac{1}{s(s+a)}$ (*ii*) $\frac{1}{s(s^2+1)}$ (*iii*) $\frac{s^2+3}{s(s^2+9)}$

Solution. (*i*) $$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt = \int_0^t e^{-at}\,dt = \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{e^{-at}}{-a}+\frac{1}{a} = \frac{1}{a}\left[1-e^{-at}\right] \qquad \textbf{Ans.}$$

(*ii*) $$L^{-1}\frac{1}{s^2+1} = \sin t$$

$$L^{-1}\frac{1}{s}\left(\frac{1}{s^2+1}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+1}\right)dt = \int_0^t \sin t\,dt = \left[-\cos t\right]_0^t = -\cos t+1$$

Ans.

(*iii*) $$L^{-1}\frac{s^2+3}{s(s^2+9)} = L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right] = L^{-1}\left[\frac{1}{s}-\frac{6}{s(s^2+9)}\right]$$

$$= 1-2\int_0^t \sin 3t\;dt = 1-\int_0^t L^{-1}\left(\frac{6}{s^2+9}\right)ds = 1+2\times\frac{1}{3}[\cos 3t]_0^t = 1+\frac{2}{3}\cos 3t-\frac{2}{3}$$

$$= \frac{2}{3}\cos 3t+\frac{1}{3} = \frac{1}{3}[2\cos 3t+1] \qquad \textbf{Ans.}$$

Exercise 13.10

Find the inverse Laplace transform of the following:

1. $\frac{1}{2s(s-3)}$ **Ans.** $\frac{1}{2}\left[\frac{e^{3t}}{3}-1\right]$ 2. $\frac{1}{s(s+2)}$ **Ans.** $\frac{1-e^{-2t}}{2}$

3. $\frac{1}{s(s^2-16)}$ **Ans.** $\frac{1}{16}[\cosh 4t-1]$ 4. $\frac{1}{s(s^2+a^2)}$ **Ans.** $\frac{1-\cos at}{a^2}$

5. $\frac{s^2+2}{s(s^2+4)}$ **Ans.** $\cos^2 t$ 6. $\frac{1}{s^2(s+1)}$ **Ans.** $t-1+e^{-t}$

7. $\frac{1}{s^3(s^2+1)}$ **Ans.** $\frac{t^2}{2}+\cos t-1$

8. $L^{-1}\frac{1}{s(s^2+1)}$ is

(*i*) $1-\cos t$ (*ii*) $1+\cos t$ (*iii*) $1-\sin t$ (*iv*) $1+\sin t$ **Ans.** (*i*)

13.24 FIRST SHIFTING PROPERTY

If $L^{-1}F(s) = f(t)$, then $L^{-1}F(s+a) = e^{-at}L^{-1}[F(s)]$

Example 34. *Find the inverse Laplace transform of*

(*i*) $\frac{1}{(s+2)^5}$ (*ii*) $\frac{s}{s^2+4s+13}$ (*iii*) $\frac{1}{9s^2+6s+1}$

Solution. (*i*) $$L^{-1}\frac{1}{s^5} = \frac{t^4}{4\,!}$$

then $$L^{-1}\frac{1}{(s+2)^5} = e^{-2t}\cdot\frac{t^4}{4\,!} \qquad \textbf{Ans.}$$

(*ii*) $$L^{-1}\left(\frac{s}{s^2+4s+13}\right) = L^{-1}\frac{s+2-2}{(s+2)^2+(3)^2} = L^{-1}\frac{s+2}{(s+2)^2+(3)^2}-L^{-1}\frac{2}{(s+2)^2+3^2}$$

$$= e^{-2t}L^{-1}\frac{s}{s^2+3^2}-e^{-2t}L^{-1}\frac{2}{3}\left(\frac{3}{s^2+3^2}\right)$$

$$= e^{-2t}\cos 3t-\frac{2}{3}e^{-2t}\sin 3t \qquad \textbf{Ans.}$$

(*iii*) $$L^{-1}\frac{1}{9s^2+6s+1} = L^{-1}\frac{1}{(3s+1)^2} = \frac{1}{9}L^{-1}\frac{1}{\left(s+\frac{1}{3}\right)^2} = \frac{1}{9}e^{-t/3}L^{-1}\frac{1}{s^2}$$

$$= \frac{1}{9}e^{-t/3}\,t = \frac{t\,e^{-t/3}}{9} \qquad \textbf{Ans.}$$

Exercise 13.11

Obtain the inverse Laplace transform of the following:

1. $\dfrac{s+8}{s^2+4s+5}$ **Ans.** $e^{-2t}(\cos t + 6\sin t)$
2. $\dfrac{s}{(s+3)^2+4}$ **Ans.** $e^{-3t}(\cos 2t - 1.5\sin 2t)$
3. $\dfrac{s}{(s+7)^4}$ **Ans.** $e^{-7t}\dfrac{t^2}{6}(3-7t)$
4. $\dfrac{s+2}{s^2-2s-8}$ **Ans.** $e^{t}(\cosh 3t + \sinh 3t)$
5. $\dfrac{s}{s^2+6s+25}$ **Ans.** $e^{-3t}\left[\cos 4t - \dfrac{3}{4}\sin 4t\right]$
6. $\dfrac{1}{2(s-1)^2+32}$ **Ans.** $\dfrac{e^t}{8}\sin 4t$
7. $\dfrac{s-4}{4(s-3)^2+16}$ **Ans.** $\dfrac{1}{4}e^{3t}\cos 2t - \dfrac{1}{8}e^{3t}\sin 2t$

13.25 SECOND SHIFTING PROPERTY

$$L^{-1}\left[e^{-as}F(s)\right] = f(t-a)\,U(t-a)$$

Example 35. *Obtain inverse Laplace transform of*

$$(i)\ \frac{e^{-\pi s}}{(s+3)} \qquad (ii)\ \frac{e^{-s}}{(s+1)^3}$$

Solution. (*i*) $L^{-1}\dfrac{1}{s+3} = e^{-3t}$

$$L^{-1}\frac{e^{-\pi s}}{s+3} = e^{-3(t-\pi)}\,U(t-\pi) \qquad \textbf{Ans.}$$

(*ii*) $L^{-1}\dfrac{1}{s^3} = \dfrac{t^2}{2!}$

$$L^{-1}\frac{1}{(s+1)^3} = e^{-t}\frac{t^2}{2!}$$

$$L^{-1}\frac{e^{-s}}{(s+1)^3} = e^{-(t-1)}\cdot\frac{(t-1)^2}{2!}\cdot U(t-1) \qquad \textbf{Ans.}$$

Example 36. *Find the inverse Laplace transform of*

$$\frac{s\,e^{-s/2}+\pi\,e^{-s}}{s^2+\pi^2}$$

in terms of unit step functions.

Solution.

$$L^{-1}\frac{\pi}{s^2+\pi^2} = \sin\pi t$$

$$L^{-1}\left[e^{-s}\frac{\pi}{s^2+\pi^2}\right] = \sin\pi(t-1)\cdot u(t-1)$$

$$= -\sin(\pi t)\cdot u(t-1) \qquad \ldots(1)$$

and

$$L^{-1}\frac{s}{s^2+\pi^2} = \cos\pi t$$

$$L^{-1}\left[e^{-s/2}\frac{s}{s^2+\pi^2}\right] = \cos\pi\left(t-\frac{1}{2}\right)\cdot u\left(t-\frac{1}{2}\right)$$

$$= \sin\pi t\cdot u\left(t-\frac{1}{2}\right) \qquad \ldots(2)$$

On adding (1) and (2), we get

$$L^{-1}\left[\frac{e^{-s/2}\,s+e^{-s}\cdot\pi}{s^2+\pi^2}\right] = \sin(\pi t)\cdot u\left(t-\frac{1}{2}\right) - \sin(\pi t)\cdot u(t-1)$$

$$= \sin\pi t\left[u\left(t-\frac{1}{2}\right) - u(t-1)\right] \qquad \textbf{Ans.}$$

Exercise 13.12

Obtain inverse Laplace transform of the following:

1. $\dfrac{e^{-s}}{(s+2)^3}$ **Ans.** $e^{-(t-2)}\dfrac{(t-2)^2}{2}U(t-2)$
2. $\dfrac{e^{-2s}}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-(t-2)}\{1-\cos(t-2)\}\,U(t-2)$
3. $\dfrac{e^{-s}}{\sqrt{s+1}}$ **Ans.** $\dfrac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}\cdot U(t-1)$
4. $\dfrac{e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}}{s^2+1}$ **Ans.** $\cos t\left[U\left(t-\dfrac{3\pi}{2}\right)-U\left(t-\dfrac{\pi}{2}\right)\right]$
5. $\dfrac{e^{-4s}(s+2)}{s^2+4s+5}$ **Ans.** $e^{-2(t-4)}\cos(t-4)\,U(t-4)$
6. $\dfrac{e^{-as}}{s^2}$ **Ans.** $f(t) = t-a$ when $t>a$; $= 0$ when $t<a$
7. $\dfrac{e^{-\pi s}}{s^2+1}$ **Ans.** $-\sin t\cdot u(t-\pi)$

Tick (√) the correct answers:

8. (*a*) The inverse Laplace transform of $(e^{-3s})/s^3$, is

(*i*) $(t-3)\,u_3(t)$ (*ii*) $(t-3)^2\,u_3(t)$ (*iii*) $(t-3)^2\,u_3(t)$ (*iv*) $(t+3)\,u_3(t)$. **Ans.** (*iv*)

(*b*) If Laplace transform of a function $f(t)$ equals $(e^{-2s}-e^{-s})/s$, then

(*i*) $f(t) = 1,\ t>1$;
(*ii*) $f(t) = 1$, when $1<t<2$, and 0 otherwise ;
(*iii*) $f(t) = -1$, when $1<t<3$, and 0 otherwise ;
(*iv*) $f(t) = -1$, when $1<t<2$, and 0 otherwise. **Ans.** (*iv*)

(*c*) The Laplace inverse $L^{-1}\left[\dfrac{2}{s}(e^{-2s}-e^{-4s})\right]$ equals

(*i*) 2, if $0<t<4$; 0 otherwise, (*ii*) 2, if $t>0$
(*iii*) 2, if $0<t<2$; 0 otherwise, (*iv*) 2, if $2<t<4$; 0 otherwise **Ans.** (*iv*)

(*d*) The Laplace transform of $t\,u_2(t)$ is

(*i*) $\left(\dfrac{1}{s^2}+\dfrac{2}{2}\right)e^{-2s}$ (*ii*) $\dfrac{1}{s^2}e^{-2s}$ (*iii*) $\left(\dfrac{1}{s^2}-\dfrac{2}{s}\right)e^{-2s}$ (*iv*) $\dfrac{1}{s^2}e^{-2s}$ **Ans.** (*i*)

(*e*) The inverse Laplace transform of $\dfrac{Ke^{-as}}{S^2+k^2}$ is

(*i*) $\sin kt$ (*ii*) $\cos kt$ (*iii*) $u(t-a)\sin kt$ (*iv*) none of these. **Ans.** (*iv*)

(*f*) Inverse Laplace's transform of 1 is:

(*i*) 1 (*ii*) $\delta(t)$ (*iii*) $\delta(t-1)$ (*iv*) $u(t)$. **Ans.** (*ii*)

13.26 INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,L^{-1}[F(s)] = -t\,f(t) \qquad \text{or} \qquad L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Example 37. *Find inverse Laplace transform of* $\tan^{-1}\dfrac{1}{s}$.

Solution.
$$L^{-1}\left(\tan^{-1}\frac{1}{s}\right) = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\frac{1}{s}\right]$$
$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(-\frac{1}{s^2}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{1}{1+s^2}\right]$$
$$= \frac{\sin t}{t} \qquad \textbf{Ans.}$$

Example 38. *Obtain the inverse Laplace transform of* $\log\dfrac{s^2-1}{s^2}$.

Solution.
$$L^{-1}\left[\log\frac{s^2-1}{s^2}\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\frac{s^2-1}{s^2}\right]$$
$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\{\log(s^2-1) - 2\log s\}\right] = -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1} - \frac{2}{s}\right] = -\frac{1}{t}[2\cosh t - 2]$$
$$= \frac{2}{t}[1-\cosh t] \qquad \textbf{Ans.}$$

Example 39. *Find* $L^{-1}\left[\cot^{-1}(1+s)\right]$.

Solution.
$$L^{-1}\left[\cot^{-1}(1+s)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(1+s)\right]$$
$$= -\frac{1}{t}L^{-1}\left[\frac{-1}{1+(s+1)^2}\right] = \frac{1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$
$$= \frac{1}{t}e^{-t}\sin t \qquad \textbf{Ans.}$$

Exercise 13.13

Obtain inverse Laplace transform of the following:

1. $\log\left(1+\dfrac{\omega^2}{s^2}\right)$ **Ans.** $-\dfrac{2}{t}\cos\omega t + 2$
2. $\dfrac{s}{1+s^2+s^4}$ **Ans.** $\dfrac{2}{\sqrt{3}}\sin\dfrac{\sqrt{3}}{2}t\,\sinh\dfrac{t}{2}$
3. $\dfrac{s+1}{(s^2+6s+13)^2}$ **Ans.** $\dfrac{e^{-3t}}{8}[2t\sin 2t + 2t\cos 2t - \sin 2t]$
4. $\dfrac{s}{(s^2+a^2)^2}$ **Ans.** $\dfrac{t\sin at}{2a}$
5. $s\log\dfrac{s}{\sqrt{s^2+1}} + \cot^{-1}s$ **Ans.** $\dfrac{1-\cos t}{t^2}$
6. $\dfrac{1}{2}\log\left\{\dfrac{s^2+b^2}{(s-a)^2}\right\}$ **Ans.** $\dfrac{e^{-at}-\cos bt}{t}$
7. $\tan^{-1}(s+1)$ **Ans.** $-\dfrac{1}{t}e^{-t}\sin t$

13.27 INVERSE LAPLACE TRANSFORM OF INTEGRALS

$$L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t} = \frac{1}{t}L^{-1}[F(s)] \qquad \text{or} \qquad L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right].$$

Example 40. *Obtain* $L^{-1}\dfrac{2s}{(s^2+1)^2}$. *(A.M.I.E.T.E., Winter 1997)*

Solution.
$$L^{-1}\frac{2s}{(s^2+1)^2} = t\,L^{-1}\int_s^\infty \frac{2s\,ds}{(s^2+1)^2} = t\,L^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty = t\,L^{-1}\left[-0+\frac{1}{s^2+1}\right]$$
$$= t\sin t \qquad \textbf{Ans.}$$

13.28 PARTIAL FRACTIONS METHOD

Example 41. *Find the inverse transforms of* $\dfrac{1}{s^2-5s+6}$.

Solution. Let us convert the given function into partial fractions.
$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$
$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t} \qquad \textbf{Ans.}$$

Example 42. *Find the inverse Laplace transforms of*
$$\frac{s-1}{s^2-6s+25}$$

Solution.
$$L^{-1}\left(\frac{s-1}{s^2-6s+25}\right) = L^{-1}\left[\frac{s-1}{(s-3)^2+(4)^2}\right] = L^{-1}\left[\frac{s-3+2}{(s-3)^2+(4)^2}\right]$$
$$= L^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{4}{(s-3)^2+(4)^2}\right]$$
$$= e^{3t}\cos 4t + \frac{1}{2}e^{3t}\sin 4t. \qquad \textbf{Ans.}$$

Example 43. *Find the inverse Laplace transforms of*
$$\frac{s+4}{s(s-1)(s^2+4)}.$$

Solution. Let us first resolve $\dfrac{s+4}{s(s-1)(s^2+4)}$ into partial fractions.
$$\frac{s+4}{s(s-1)(s^2+4)} \equiv \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4} \qquad \text{... (1)}$$
$$s+4 \equiv A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)s(s-1)$$

Putting $s = 0$, we get $4 = -4A$ or $A = -1$

Putting $s = 1$, we get $5 = B\cdot 1\cdot(1+4)$ or $B = 1$

Equating the coefficients of s^3 on both sides of (1), we have
$$0 = A + B + C \quad \text{or} \quad 0 = -1 + 1 + C \quad \text{or} \quad C = 0.$$
Equating the coefficients of s on both sides of (1), we get
$$1 = 4A + 4B - D \quad \text{or} \quad 1 = -4 + 4 - D \quad \text{or} \quad D = -1.$$
On putting the values of A, B, C, D in (1), we get
$$\frac{s+4}{s(s-1)(s^2+4)} = -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$\therefore \quad L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right] = L^{-1}\left[-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}\right]$$

$$= -L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= -1+e^t-\frac{1}{2}\sin 2t. \qquad \textbf{Ans.}$$

Example 44. *Find the Laplace inverse of*

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}.$$

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{a^2}{a^2-b^2}\cdot\frac{1}{s^2+a^2}-\frac{b^2}{a^2-b^2}\cdot\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}L^{-1}\left[\frac{a^2}{s^2+a^2}-\frac{b^2}{s^2+b^2}\right] = \frac{1}{a^2-b^2}\left[a^2\left(\frac{1}{a}\sin at\right)-b^2\left(\frac{1}{b}\sin bt\right)\right]$$

$$= \frac{1}{a^2-b^2}[a\sin at-b\sin bt]. \qquad \textbf{Ans.}$$

Exercise 13.14

Find the inverse transforms of:

1. $\dfrac{s^2+2s+6}{s^3}$ **Ans.** $1+2t+3t^2$
2. $\dfrac{1}{s^2-7s+12}$ **Ans.** $e^{4t}-e^{3t}$
3. $\dfrac{s+2}{s^2-4s+13}$ **Ans.** $e^{2t}\cos 3t+\dfrac{4}{3}e^{2t}\sin 3t$
4. $\dfrac{3s+1}{(s-1)(s^2+1)}$ **Ans.** $e^t-2\cos t+\sin t$
5. $\dfrac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ **Ans.** $2e^{-t}+5e^{2t}-\dfrac{3}{2}e^{t/2}$
6. $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$ **Ans.** $\dfrac{1}{2}e^t-e^{2t}+\dfrac{5}{2}e^{3t}$
7. $\dfrac{s-4}{(s-4)^2+9}$ **Ans.** $e^{4t}\cos 3t$
8. $\dfrac{16}{(s^2+2s+5)^2}$ **Ans.** $e^{-t}(\sin 2t-2t\cos 2t)$
9. $\dfrac{1}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-t}(1-\cos t)$
10. $\dfrac{1}{(s-2)(s^2+1)}$ **Ans.** $\dfrac{1}{5}e^{2t}-\dfrac{1}{5}\cos t-\dfrac{2}{5}\sin t$
11. $\dfrac{s^2-6s+7}{(s^2-4s+5)^2}$ **Ans.** $e^{2t}[t\cos t-\sin t]$

13.29 INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$L\left\{\int_0^t f_1(x)*f_2(t-x)\,dx\right\} = F_1\cdot(s)\cdot F_2(s) \quad\text{or}\quad \int_0^t f_1(x)\cdot f_2(t-x)\,dx = L^{-1}\,F_1(s)\cdot F_2(s)$$

Example 45. *Using the convolution theorem, find*

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\},\quad a\neq b.$$

Solution. We have

$$L(\cos at) = \frac{s}{s^2+a^2} \quad\text{and}\quad L(\cos bt) = \frac{s}{s^2+b^2}$$

Hence by the convolution theorem

$$L\left\{\int_0^t \cos ax\cos b(t-x)\,dx\right\} = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Therefore,

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos ax\cos b(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t\{\cos(ax+bt-bx)+\cos(ax-bt+bx)\}\,dx$$

$$= \frac{1}{2}\int_0^t \cos[(a-b)x+bt]\,dx+\frac{1}{2}\int_0^t \cos[(a+b)x-bt]\,dx$$

$$= \left[\frac{\sin[(a-b)x+bt]}{2(a-b)}\right]_0^t+\left[\frac{\sin[(a+b)x-bt]}{2(a+b)}\right]_0^t$$

$$= \frac{\sin at-\sin bt}{2(a-b)}+\frac{\sin at+\sin bt}{2(a+b)}$$

$$= \frac{a\sin at-b\sin bt}{a^2-b^2} \qquad \textbf{Ans.}$$

Example 46. *Obtain* $L^{-1}\dfrac{1}{s(s^2+a^2)}$.

Solution. $L^{-1}\dfrac{1}{s} = 1$ and $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{\sin at}{a}$.

Hence by the convolution theorem

$$L\int_0^t\left\{1\cdot\frac{\sin a(t-x)}{a}\,dx\right\} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2+a^2}\right)$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t\frac{\sin a(t-x)}{a}\,dx = \left[\frac{-\cos(at-ax)}{-a^2}\right]_0^t$$

$$= \frac{1}{a^2}[1-\cos at] \qquad \textbf{Ans.}$$

Exercise 13.15

Obtain the inverse Laplace transform by convolution.

1. $\dfrac{s^2}{(s^2+a^2)^2}$ **Ans.** $\dfrac{1}{2}t\cos at+\dfrac{1}{2a}\sin at$
2. $\dfrac{1}{(s^2+1)^3}$ **Ans.** $\dfrac{1}{8}\{(3-t^2)\sin t-3t\cos t\}$
3. $\dfrac{s}{(s^2+a^2)^2}$ **Ans.** $\dfrac{t\sin at}{2a}$;
4. $\dfrac{1}{s^2(s^2-a^2)}$ **Ans.** $\dfrac{1}{a^3}[-at+\sin h\,at]$
5. $\dfrac{1}{(s+1)(s^2+1)}$; **Ans.** $\dfrac{1}{2}(\cos t-\sin t-e^{-t})$

13.30. SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants.

The method will be clear from the following examples:

Example 47. *Using Laplace transforms, find the solution of the initial value problem*

$$y''-4y'+4y = 64\sin 2t$$

$$y(0) = 0,\ y'(0) = 1.$$

Solution. $$y'' - 4y' + 4y = 64 \sin 2t \qquad \text{... (1)}$$

$$y(0) = 0,\ y'(0) = 1$$

Taking Laplace transform of both sides of (1), we have

$$[s^2 \bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = \frac{64 \times 2}{s^2+4} \qquad \text{... (2)}$$

On putting the values of $y(0)$ and $y'(0)$ in (2), we get

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = \frac{128}{s^2+4}$$

$$(s^2 - 4s + 4)\bar{y} = 1 + \frac{128}{s^2+4}, \quad \text{or} \quad (s-2)^2 \bar{y} = 1 + \frac{128}{s^2+4}$$

$$\bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s-2)^2(s^2+4)} = \frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2+4}$$

$$y = L^{-1}\left[-\frac{8}{s-2} + \frac{17}{(s-2)^2} + \frac{8s}{s^2+4}\right]$$

$$y = -8e^{2t} + 17t\,e^{2t} + 8\cos 2t \qquad \textbf{Ans.}$$

Example 48. *Using the Laplace transfors, find the solution of the initial value problem*

$$y'' + 25y = 10\cos 5t$$

$$y(0) = 2, \quad y'(0) = 0.$$

Solution. Taking Laplace transform of the given differential equation, we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 25\bar{y} = 10\frac{s}{s^2+25}$$

$$s^2\bar{y} - 2s + 25\bar{y} = \frac{10s}{s^2+25}$$

$$(s^2+25)\bar{y} = 2s + \frac{10s}{s^2+25}$$

$$\bar{y} = \frac{2s}{s^2+25} + \frac{10s}{(s^2+25)^2}$$

$$y = L^{-1}\left[\frac{2s}{s^2+25} + \frac{10s}{(s^2+25)^2}\right] = 2\cos 5t + L^{-1}\left[\frac{10s}{(s^2+25)^2}\right]$$

$$= 2\cos 5t + L^{-1}\frac{d}{ds}\left[\frac{-5}{(s^2+25)}\right]$$

$$= 2\cos 5t + t\sin 5t \qquad \textbf{Ans.}$$

Example 49. *Applying convolution, solve the following initial value problem*

$$y'' + y = \sin 3t,$$

$$y(0) = 0, \quad y'(0) = 0.$$

Solution. $y'' + y = \sin 3t$

Taking Laplace transform of both the sides, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = \frac{3}{s^2+9} \qquad \text{...(1)}$$

On putting the values of $y(0), y'(0)$ in (1) we get

$$s^2\bar{y} + \bar{y} = \frac{3}{s^2+9} \quad \text{or} \quad (s^2+1)\bar{y} = \frac{3}{s^2+9}$$

$$\bar{y} = \frac{3}{(s^2+1)(s^2+9)} = \frac{3}{8}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

Taking the inversion transform we get

$$y = \frac{3}{8}L^{-1}\frac{1}{s^2+1} - \frac{3}{8}L^{-1}\frac{1}{s^2+9}$$

$$y = \frac{3}{8}\sin t - \frac{3}{8}\times\frac{1}{3}\sin 3t = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t \qquad \textbf{Ans.}$$

Example 50. *Solve* $[t D^2 + (1-2t) D - 2]\,y = 0,$ *where* $y(0) = 1,\ y'(0) = 2.$

(*R.G.P.V. June, 2002*)

Solution. Here, $t D^2 y + (1-2t) Dy - 2y = 0 \quad \Rightarrow \quad t y'' + y' - 2t y' - 2y = 0$

Taking Laplace transform of given differential equation, we get

$$L(t y'') + L(y') - 2L(t y') - 2L(y) = 0 \Rightarrow -\frac{d}{ds}L\{y''\} + L\{y'\} + 2\frac{d}{ds}L(y') - 2L(y) = 0$$

$$-\frac{d}{ds}[s^2\bar{y} - s\,y(0) - y'(0)] + [s\bar{y} - y(0)] + 2\frac{d}{ds}[s\bar{y} - y(0)] - 2\bar{y} = 0$$

Putting the values of $y(0)$ and $y'(0)$, we get

$$-\frac{d}{ds}(s^2\bar{y} - s - 2) + (s\bar{y} - 1) + 2\frac{d}{ds}(s\bar{y} - 1) - 2\bar{y} = 0 \qquad [\because y(0) = 1,\ y'(0) = 2]$$

$$\Rightarrow -\frac{s^2 d\bar{y}}{ds} - 2s\bar{y} + 1 + s\bar{y} - 1 + 2\left(s\frac{d\bar{y}}{ds} + \bar{y}\right) - 2\bar{y} = 0 \Rightarrow -(s^2 - 2s)\frac{d\bar{y}}{ds} - s\bar{y} = 0$$

$$\Rightarrow -\frac{d\bar{y}}{\bar{y}} + \frac{1}{s-2}\,ds = 0 \qquad \text{(Separating the variables)}$$

$$\Rightarrow \int \frac{d\bar{y}}{\bar{y}} + \int \frac{ds}{s-2} = 0 \quad \Rightarrow \log\bar{y} + \log(s-2) = \log C$$

$$\Rightarrow \bar{y}(s-2) = C \quad \Rightarrow \bar{y} = \frac{C}{s-2} \quad \Rightarrow y = C L^{-1}\left\{\frac{1}{s-2}\right\} \Rightarrow y = C e^{2t} \quad \text{...(1)}$$

Putting $y(0) = 1$ in (1), we get $\quad 1 = C e^0 \quad \Rightarrow \quad C = 1$

Putting $C = 1$ in (1), we get $y = e^{2t}$

This is the required solution. **Ans.**

Example 51. *Using Laplace transform technique solve the following initial value problem*

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin t, \qquad \textit{where } y(0) = y'(0) = 0.$$

Solution. $$y'' + 2y' + 2y = 5\sin t$$

$$y(0) = y'(0) = 0$$

Taking the Laplace Transform of both sides, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 2\bar{y} = 5\times\frac{1}{s^2+1} \qquad \text{...(1)}$$

On substituting the values of $y(0)$, and $y'(0)$ in (1), we get

$$s^2\bar{y} + 2s\bar{y} + 2\bar{y} = \frac{5}{s^2+1} \quad \text{or} \quad [s^2 + 2s + 2]\bar{y} = \frac{5}{s^2+1}$$

$$\bar{y} = \frac{5}{(s^2+2s+2)(s^2+1)}$$

Resolving into partial fractions, $y = \dfrac{2s+3}{s^2+2s+2} + \dfrac{-2s+1}{s^2+1}$

Taking the inverse transform, we get

$$y = L^{-1}\left(\frac{2s+3}{s^2+2s+2}\right) + L^{-1}\left(\frac{-2s+1}{s^2+1}\right)$$

$$= L^{-1}\left[\frac{2(s+1)+1}{(s+1)^2+1}\right] + L^{-1}\left(\frac{-2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= L^{-1}\left[\frac{2(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right] - 2\cos t + \sin t$$

$$= 2e^{-t}\cos t + e^{-t}\sin t - 2\cos t + \sin t \qquad \textbf{Ans.}$$

Example 52. *Solve the initial value problem*

$$2y'' + 5y' + 2y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = 1,$$

using the Laplace transforms. (*A.M.I.E.T.E., Summer 1995*)

Solution. $2y'' + 5y' + 2y = e^{-2t}, \quad y(0) = 1, y'(0) = 1$

Taking the Laplace Transform of both sides, we get

$$2[s^2\bar{y} - sy(0) - y'(0)] + 5[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s+2} \qquad ...(1)$$

On substituting the values of $y(0)$ and $y'(0)$ in (1), we get

$$2[s^2\bar{y} - s - 1] + 5[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s+2}$$

$$[2s^2 + 5s + 2]\bar{y} - 2s - 2 - 5 = \frac{1}{s+2}$$

$$\bar{y} = \frac{1}{(s+2)(2s^2+5s+2)} + \frac{2s+7}{2s^2+5s+2} = \frac{1+2s^2+7s+4s+14}{(2s^2+5s+2)(s+2)} = \frac{2s^2+11s+15}{(2s+1)(s+2)^2}$$

$$= \frac{4/9}{2s+1} - \frac{11/9}{s+2} - \frac{1/3}{(s+2)^2} = \frac{4}{9}\,\frac{1}{2}\,\frac{1}{s+\frac{1}{2}} - \frac{11}{9}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{1}{(s+2)^2}$$

$$y = \frac{2}{9}e^{-\frac{1}{2}t} - \frac{11}{9}e^{-2t} - \frac{1}{3}te^{-2t} \qquad \textbf{Ans.}$$

Example 53. *Solve* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = e^{-x}\sin x$ *where* $y(0) = 0, \quad y'(0) = 1.$

Solution. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = e^{-x}\sin x$

Taking the Laplace Transform of both the sides, we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = L(e^{-x}\sin x)$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s+1)^2+1} \qquad ...(1)$$

On substituting the values of $y(0)$ and $y'(0)$ in (1), we get

$$(s^2\bar{y} - 1) + 2(s\bar{y}) + 5\bar{y} = \frac{1}{s^2+2s+2}$$

$$(s^2+2s+5)\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

$$\bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

On resolving the R.H.S. into partial fractions, we get

$$\bar{y} = \frac{2}{3}\,\frac{1}{s^2+2s+5} + \frac{1}{3}\,\frac{1}{s^2+2s+2}$$

On inversion, we obtain

$$y = \frac{2}{3}L^{-1}\frac{1}{s^2+2s+5} + \frac{1}{3}L^{-1}\frac{1}{s^2+2s+2}$$

or $$y = \frac{1}{3}L^{-1}\frac{2}{(s+1)^2+(2)^2} + \frac{1}{3}L^{-1}\frac{1}{(s+1)^2+(1)^2}$$

or $y = \frac{1}{3}e^{-x}\sin 2x + \frac{1}{3}e^{-x}\sin x$ or $y = \frac{1}{3}e^{-x}(\sin x + \sin 2x)$ **Ans.**

Example 54. *Using Laplace transforms, find the solution of the initial value problem*

$$y'' + 9y = 9u(t-3), \quad y(0) = y'(0) = 0$$

where $u(t-3)$ is the unit step function. (*A.M.I.E.T.E., Winter 1998*)

Solution. $y'' + 9y = 9u(t-3)$. ... (1)

Taking Laplace transform of (1), we have

$$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 9\frac{e^{-3s}}{s} \qquad ...(2)$$

Putting the values of $y(0) = 0$ and $y'(0) = 0$ in (2), we get

$$s^2\bar{y} + 9\bar{y} = \frac{9e^{-3s}}{s}$$

$$(s^2+9)\bar{y} = 9\frac{e^{-3s}}{s}$$

$$\bar{y} = \frac{9e^{-3s}}{s(s^2+9)} \quad \Rightarrow \quad y = L^{-1}\frac{9e^{-3s}}{s(s^2+9)}$$

$$L^{-1}\frac{3}{s^2+9} = \sin 3t$$

$$3L^{-3}\frac{3}{s(s^2+9)} = 3\int_0^t \sin 3t\,dt = -[\cos 3t]_0^t = 1 - \cos 3t$$

$$y = L^{-1}\frac{9e^{-3s}}{s(s^2+9)}$$

$$y = [1 - \cos 3(t-3)]\,u(t-3) \qquad \textbf{Ans.}$$

Example 55. *A resistance R in series with inductance L is connected with e.m.f. E. (t). The current i is given by*

$$L\frac{di}{dt} + Ri = E(t).$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i in terms of t. (*U.P., II Semester, Summer 2001*)

Solution. Conditions under which current i flows are $i = 0$ at $t = 0$,

$$E(t) = \begin{cases} E, & 0 < t < a \\ 0, & t > a \end{cases}$$

Given equation is $\quad L\dfrac{di}{dt} + Ri = E(t) \quad$ ...(1)

Taking Laplace transform of (1), we get

$$L\,[s\bar{i} - i(0)] + R\bar{i} = \int_0^\infty e^{-st} E(t)\,dt$$

$$L\,s\bar{i} + R\bar{i} = \int_0^\infty e^{-st} E(t)\,dt \qquad [i(0) = 0]$$

$$(L\,s + R)\,\bar{i} = \int_0^\infty e^{-st}\,.\,E\,dt = \int_0^a e^{-st} E\,dt + \int_a^\infty e^{-st} E\,dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a + 0 = \frac{E}{s}[1 - e^{-as}] = \frac{E}{s} - \frac{E}{s}e^{-as}$$

$$\bar{i} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}$$

On inversion, we obtain

$$i = L^{-1}\left[\frac{E}{s(Ls+R)}\right] - L^{-1}\left[\frac{E\,e^{-as}}{s(Ls+R)}\right] \qquad ...(2)$$

Now we have to find the value of $L^{-1}\left[\dfrac{E}{s(Ls+R)}\right]$

$$L^{-1}\left[\frac{E}{s(Ls+R)}\right] = \frac{E}{L}L^{-1}\left[\frac{1}{s\left(s+\frac{R}{L}\right)}\right] \quad \text{(Resolving into partial fractions)}$$

$$= \frac{E}{L}\frac{L}{R}L^{-1}\left[\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right] = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

and

$$L^{-1}\left[\frac{E\,e^{-as}}{s(Ls+R)}\right] = \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]u(t-a)$$

[By the second shifting theorem]

On substituting the values of the inverse transforms in (2) we get

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]u(t-a)$$

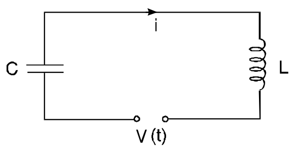
Hence

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] \quad \text{for } 0 < t < a, \quad [u(t-a) = 0]$$

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left\{1 - e^{-\frac{R}{L}(t-a)}\right\} \qquad \text{for } t > a \quad [u(t-a) = 1]$$

$$= \frac{E}{R}\left[e^{-\frac{R}{L}(t-a)} - e^{-\frac{R}{L}t}\right] = \frac{E}{R}e^{-\frac{R}{L}t}\left[e^{\frac{Ra}{L}} - 1\right] \qquad \textbf{Ans.}$$

Example 56. *Using the Laplace transform, find the current $i(t)$ in the LC-circuit. Assuming $L = 1$ henry, $C = 1$ farad, zero initial current and charge on the capacitor, and*



$$v(t) = t \text{ when } 0 < t < 1$$
$$= 0 \text{ otherwise.}$$

Solution. The differential equation for L and C circuit is given by

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \qquad ...(1)$$

Putting $L = 1, C = 1, E = v(t)$ in (1), we get

$$\frac{d^2q}{dt^2} + q = v(t) \qquad ...(2)$$

Taking Laplace Transform of (2), we have

$$s^2\bar{q} - sq(0) - q'(0) + \bar{q} = \int_0^\infty v(t)\,e^{-st}\,dt$$

Substituting $q(0) = 0, i(0) = q'(0) = 0$, we get

$$s^2\bar{q} + \bar{q} = \int_0^1 t\,e^{-st}\,dt + \int_1^\infty 0\,e^{-st}\,dt$$

$$(s^2+1)\,\bar{q} = \left[t\,\frac{e^{-st}}{-s}\right]_0^t - \int_0^1 \frac{e^{-st}}{-s}\,dt = \frac{e^{-s}}{-s} - \left[\frac{e^{-st}}{s^2}\right]_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$\bar{q} = \frac{1}{s^2+1}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$\bar{q} = \frac{-e^{-s}}{s(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2(s^2+1)}$$

Taking inverse Laplace transform, we get

$$q = L^{-1}\frac{-e^{-s}}{s(s^2+1)} - L^{-1}\frac{e^{-s}}{s^2(s^2+1)} + L^{-1}\frac{1}{s^2(s^2+1)} \qquad ...(3)$$

We know that

$$L^{-1}[\,e^{-as}f(s)\,] = f(t-a)\,u(t-a)$$

$$L^{-1}\frac{1}{s(s^2+1)} = \int_0^t \sin t\,dt = [\,-\cos t\,]_0^t = 1 - \cos t$$

$$L^{-1}\frac{1}{s^2(s^2+1)} = \int_0^t (1 - \cos t)\,dt = t - \sin t$$

In view of this, we have

$$L^{-1}\left[\frac{-e^{-s}}{s(s^2+1)}\right] = -[1 - \cos(t-1)]\,u(t-1)$$

$$L^{-1}\frac{e^{-s}}{s^2(s^2+1)} = [(t-1) - \sin(t-1)]\,u(t-1)$$

Putting in (3) we get

$$q = -[1 - \cos(t-1)]\,u(t-1) - [(t-1) - \sin(t-1)]\,u(t-1) + t - \sin t \qquad \textbf{Ans.}$$

EXERCISE 13.16

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} + y = 0$, where $y = 1$ and $\dfrac{dy}{dx} = -1$ at $x = 0$. $\qquad$ **Ans.** $y = \cos x - \sin x$

2. $\dfrac{d^2 y}{dx^2} - 4y = 0$, where $y = 0$ and $\dfrac{dy}{dx} = -6$ at $x = 0$. **Ans.** $y = -\dfrac{3}{2} e^{2x} + \dfrac{3}{2} e^{-2x}$

3. $\dfrac{d^2 y}{dx^2} + y = 0$, where $y = 1, \dfrac{dy}{dx} = 1$ at $x = 0$. **Ans.** $y = \sin x + \cos x$

4. $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$, where $y = 2$, $\dfrac{dy}{dx} = -4$ at $x = 0$. **Ans.** $y = e^{-x}(2\cos 2x - \sin 2x)$

5. $\dfrac{d^3 y}{dx^3} + 2\dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$, given $y = \dfrac{dy}{dx} = 0$, $\dfrac{d^2 y}{dx^2} = 6$ at $x = 0$. **Ans.** $y = e^x - 3e^{-x} + 2e^{-2x}$

6. $\dfrac{d^2 y}{dx^2} + y = 3\cos 2x$, where $y = \dfrac{dy}{dx} = 0$ at $x = 0$. **Ans.** $y = \cos x - \cos 2x$.

7. $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} - 2y = 1 - 2x$, given $y = 0$, $\dfrac{dy}{dx} = 4$ at $x = 0$. **Ans.** $y = e^x - e^{-2x} + x$

8. $\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4e^{2x}$, given $y = -3$, and $\dfrac{dy}{dx} = 5$ at $x = 0$. **Ans.** $y = -7e^x + 4e^{2x} + 4x e^{2x}$

9. $\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x + e^{2x}$, where $y = 1, \dfrac{dy}{dx} = -1$ at $x = 0$. **Ans.** $y = 3 + 2x + \dfrac{1}{2} e^{3x} - 2e^{2x} - \dfrac{1}{2} e^x$.

10. $\dfrac{d^3 y}{dx^3} + 2\dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$, where $y = 1, \dfrac{dy}{dx} = 2, \dfrac{d^2 y}{dx^2} = 2$ at $x = 0$. **Ans.** $\dfrac{5}{3} e^x - e^{-x} + \dfrac{1}{3} e^{-2x}$.

11. $(D^2 - D - 2)x = 20\sin 2t$, $x_0 = -1, x_1 = 2$ **Ans.** $x = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$

12. $(D^3 + D^2)x = 6t^2 + 4, x(0) = 0, x'(0) = 2, x''(0) = 0$ **Ans.** $x = \dfrac{1}{2} t^4 - 2t^3 + 8t^2 - 16t + 16 - 16e^{-t}$

13. $\dfrac{d^2 x}{dt^2} - 2\dfrac{dx}{dt} + x = e^t$, where $x(0) = 2$, $\dfrac{dx}{dt} = -1$ at $t = 0$ **Ans.** $x = 2e^t - 3te^t + \dfrac{1}{2} t^2 e^t$

14. $(D^2 + n^2)x = a\sin(nt + \alpha)$ where $x = Dx = 0$ at $t = 0$.

Ans. $x = \text{an}\cos\alpha\,(\sin nt - nt\cos nt) + \dfrac{a\sin 2\alpha}{2n}(t\sin nt)$

15. $y'' + 2y' + y = te^{-t}$ if $y(0) = 1$, $y'(0) = -2$. **Ans.** $y = \left(1 - t + \dfrac{t^3}{6}\right)e^{-t}$

16. $\dfrac{d^2 y}{dx^2} + y = x\cos 2x$, where $y = \dfrac{dy}{dx} = 0$ at $x = 0$. **Ans.** $y = \dfrac{4}{9}\sin 2x - \dfrac{5}{9}\sin x - \dfrac{x}{3}\cos 2x$

17. $\dfrac{d^3 y}{dx^3} - 3\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} - y = x^2 e^{2x}$, where $y = 1, \dfrac{dy}{dx} = 0, \dfrac{d^2 y}{dx^2} = -2$ at $x = 0$.

Ans. $y = e^{2x}(x^2 - 6x + 12) - e^x(15x^2 + 7x + 11)$

18. $y'' + 4y' + 3y = t, t > 0$; given that $y(0) = 0$ and $y'(0) = 1$. **Ans.** $y = -\dfrac{4}{9} + \dfrac{t}{6} + e^{-t} - \dfrac{5}{9} e^{-3t}$

19. $y'' + 2y = r(t), y(0) = 0, y'(0) = 0$ where $r(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \end{cases}$ **Ans.** $y = \dfrac{1}{2} - \dfrac{1}{2}\cos\sqrt{2}\,t$.

20. $\dfrac{d^2 y}{dt^2} + 4y = u(t-2)$, where u is unit step function

$y(0) = 0$ and $y'(0) = 1$. **Ans.** $y = \dfrac{1}{2}\sin 2t$ for $t < 2$

21. $\dfrac{d^2 y}{dx^2} + y = u(t - \pi) - u(t - 2\pi)$, $y(0) = y'(0) = 0$ (*Nagpur 1995*)

Ans. $y = (1 + \cos t)\,u(t - \pi) - (1 - \cos t)\,u(t - 2\pi)$

22. A condenser of capicity C is charged to potential E and discharged at $t = 0$ through an inductance L and resistance R. The charge q at time t is governed by the differential equation

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

Using Laplace transforms, show that the charge q is given by

$$q = \frac{CE}{n} e^{-\mu t}[\mu\sin nt + n\cos nt] \quad \text{where } \mu = \frac{R}{2L} \text{ and } n^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

13.31 SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Simultaneous differential equations can also be solved by Laplace Transform method.

Example 57. *Solve* $\dfrac{dx}{dt} + y = 0$ *and* $\dfrac{dy}{dt} - x = 0$ *under the condition* $x(0) = 1, y(0) = 0.$

Solution.

$$x' + y = 0 \quad \text{...(1)}$$
$$y' - x = 0 \quad \text{...(2)}$$

Taking the Laplace transform of (1) and (2) we get

$$[s\bar{x} - x(0)] + \bar{y} = 0 \quad \text{...(3)}$$
$$[s\bar{y} - y(0)] - \bar{x} = 0 \quad \text{...(4)}$$

On substituting the values of $x(0)$ and $y(0)$ in (3) and (4) we get

$$s\bar{x} - 1 + \bar{y} = 0 \quad \text{...(5)}$$
$$s\bar{y} - \bar{x} = 0 \quad \text{...(6)}$$

Solving (5) and (6) for $\bar{x}$ and $\bar{y}$ we get

$$\bar{x} = \frac{s}{s^2 + 1}, \quad \bar{y} = \frac{1}{s^2 + 1}$$

On inversion, we obtain $x = L^{-1}\left(\dfrac{s}{s^2 + 1}\right), \quad y = L^{-1}\left(\dfrac{1}{s^2 + 1}\right)$

$$x = \cos t, \quad y = \sin t$$ **Ans.**

Example 58. *Using Laplace transforms, solve the differential equations*

$$(D + 1)y_1 + (D - 1)y_2 = e^{-t}$$
$$(D + 2)y_1 + (D + 1)y_2 = e^t$$

where $D = d/dt$ and $y_1(0) = 1, \ y_2(0) = 0$

Solution.

$$(D + 1)y_1 + (D - 1)y_2 = e^{-t} \quad \text{...(1)}$$
$$(D + 2)y_1 + (D + 1)y_2 = e^t \quad \text{...(2)}$$

Multiply (1) by $(D + 1)$ and (2) by $(D - 1)$ we get

$$(D + 1)^2 y_1 + (D^2 - 1)y_2 = (D + 1)e^{-t} \quad \text{...(3)}$$
$$(D - 1)(D + 2)y_1 + (D^2 - 1)y_2 = (D - 1)e^t \quad \text{...(4)}$$

Subtracting (4) from (3) we get

$$(D^2 + 2D + 1 - D^2 - D + 2)y_1 = (-e^{-t} + e^{-t}) - (e^t - e^t)$$

or $(D + 3)y_1 = 0$ or $Dy_1 + 3y_1 = 0$

Taking Laplace transform we have $s\bar{y}_1 - y_1(0) + 3\bar{y}_1 = 0$

$$(s + 3)\bar{y}_1 = 1 \quad \text{or} \quad \bar{y}_1 = \frac{1}{s + 3} \quad \text{or} \quad y_1 = e^{-3t}$$

Putting the value of y_1 in (1) we get

$$(D+1)\,e^{-3t}+(D-1)\,y_2 = e^{-t}$$

$$-3\,e^{-3t}+e^{-3t}+(D-1)\,y_2 = e^{-t}$$

$$(D-1)\,y_2 = e^{-t}+2\,e^{-3t} \quad \text{or} \quad Dy_2 - y_2 = e^{-t}+2\,e^{-3t}$$

Taking Laplace transform we get

$$s\,\bar{y}_2 - y_2\,(0) - \bar{y}_2 = \frac{1}{s+1}+\frac{2}{s+3}$$

$$(s-1)\,\bar{y}_2 = \frac{1}{s+1}+\frac{2}{s+3}$$

$$\bar{y}_2 = \frac{1}{s^2-1}+\frac{2}{s^2+2\,s-3}$$

$$y_2 = L^{-1}\left[\frac{1}{s^2-1}+\frac{2}{(s+1)^2-(2)^2}\right]$$

$$y_2 = \sinh t + e^{-t}\sinh 2\,t$$

$$y_1 = e^{-3t} \quad \text{and} \quad y_2 = \sinh t + e^{-t}\sinh 2\,t \qquad \textbf{Ans.}$$

Example 59. *Solve* $\dfrac{dx}{dt} - y = e^t,\ \dfrac{dy}{dt} + x = \sin t$

given $x\,(0) = 1,\ y\,(0) = 0.$ *(A.M.I.E.T.E., Summer 1997)*

Solution.

$$x' - y = e^t \qquad \ldots(1)$$

$$y' + x = \sin t \qquad \ldots(2)$$

Taking the Laplace Transform of (1) and (2), we get

$$[s\,\bar{x} - x\,(0)] - \bar{y} = \frac{1}{s-1} \qquad \ldots(3)$$

$$[s\,\bar{y} - y\,(0)] + \bar{x} = \frac{1}{s^2+1} \qquad \ldots(4)$$

On substituting the values of $x\,(0)$ and $y\,(0)$ in (3) and (4) we get

$$s\,\bar{x} - 1 - \bar{y} = \frac{1}{s-1} \qquad \ldots(5)$$

$$s\,\bar{y} + x = \frac{1}{s^2+1} \qquad \ldots(6)$$

On solving (5) and (6), we get

$$\bar{x} = \frac{s^4+s^2+s-1}{(s-1)\,(s^2+1)^2} = \frac{1}{2}\,\frac{1}{s-1}+\frac{1}{2}\,\frac{s+1}{s^2+1}+\frac{1}{(s^2+1)^2} \qquad \ldots(7)$$

$$\bar{y} = \frac{-s^3+s^2-2\,s}{(s-1)\,(s^2+1)^2} = -\frac{1}{2}\,\frac{1}{s-1}+\frac{1}{2}\,\frac{s-1}{(s^2+1)}+\frac{s}{(s^2+1)^2} \qquad \ldots(8)$$

On inversion of (7), we obtain

$$x = \frac{1}{2}L^{-1}\frac{1}{s-1}+\frac{1}{2}L^{-1}\frac{s}{s^2+1}\ \frac{1}{2}L^{-1}+\frac{1}{s^2+1}+L^{-1}\frac{1}{(s^2+1)^2}$$

$$= \frac{1}{2}\,e^t+\frac{1}{2}\cos t+\frac{1}{2}\sin t+\frac{1}{2}\,(\sin t - t\cos t) = \frac{1}{2}\,[e^t+\cos t+2\,\sin t - t\cos t]$$

On inversion of (8), we get

$$y = -\frac{1}{2}L^{-1}\frac{1}{s-1}+\frac{1}{2}\ L^{-1}\frac{s}{s^2+1}-\frac{1}{2}L^{-1}\frac{1}{s^2+1}+L^{-1}\frac{s}{(s^2+1)^2}$$

$$y = -\frac{1}{2}\,e^t+\frac{1}{2}\cos t-\frac{1}{2}\sin t+\frac{1}{2}\,t\sin t$$

$$y = \frac{1}{2}\,[-e^t-\sin t+\cos t+t\sin t] \qquad \textbf{Ans.}$$

Example 60. *Using the Laplace transforms, solve the initial value problem*

$$y_1'' = y_1 + 3y_2$$

$$y_2'' = 4y_1 - 4e^t$$

$$y_1\,(0) = 2,\ y_1'\,(0) = 3,\ \ y_2\,(0) = 1, y_2'\,(0) = 2$$ *(A.M.I.E.T.E., Winter 1996)*

Solution.

$$y_1'' = y_1 + 3y_2 \qquad \ldots(1)$$

$$y_2'' = 4y_1 - 4e^t \qquad \ldots(2)$$

Taking the Laplace transform of (1) and (2), we get

$$s^2\,\bar{y}_1 - sy_1(0) - y_1'(0) = \bar{y}_1 + 3\,\bar{y}_2 \qquad \ldots(3)$$

$$s^2\,\bar{y}_2 - sy_2\,(0) - y_2'\,(0) = 4\,\bar{y}_1 - \frac{4}{s-1} \qquad \ldots(4)$$

Putting the values of $y_1\,(0)$, $y'\,(0)$, $y_2\,(0)$, $y_2'\,(0)$ in (3) and (4), we get

$$s^2\,\bar{y}_1 - 2s - 3 = \bar{y}_1 + 3\bar{y}_2 \quad \text{or} \quad (s^2-l)\bar{y}_1 - 3\bar{y}_2 - = 2s+3 \qquad \ldots(5)$$

$$s^2\,\bar{y}_2 - s - 2 = 4\bar{y}_1 - \frac{4}{s-1} \quad \text{or} \quad 4\bar{y}_1 - s\bar{y}_2 = \frac{4}{s-1} - s - 2 \qquad \ldots(6)$$

On solving (5) and (6), we get

$$\bar{y}_1 = \frac{(2s-3)\,(s^2+3)(s+2)}{(s-1)(s^2+3)(s^2-4)} = \frac{2s-3}{(s-1)(s-2)} = \frac{1}{s-1}+\frac{1}{s-2}$$

$$y_1 = e^t + e^{2t}$$

$$\bar{y}_2 = \frac{(s+2)(s^2+3)}{(s^2+3)(s^2-4)} = \frac{1}{s-2}, \quad \Rightarrow \quad y_2 = e^{2t} \qquad \textbf{Ans.}$$

EXERCISE 13.17

Solve the following :

1. $\dfrac{dx}{dt} + 4\,y = 0,\ \ \dfrac{dy}{dt} - 9\,x = 0.$ Given $x = 2$ and $y = 1$ at $t = 0$.

Ans. $x = -\dfrac{2}{3}\sin 6\,t + 2\cos 6\,t,\ \ y = \cos 6\,t + 3\sin 6\,t$

2. $4\,\dfrac{dy}{dt} + \dfrac{dx}{dt} + 3\,y = 0,\ \ \dfrac{3\,dx}{dt} + 2\,x + \dfrac{dy}{dt} = 1$

under the condition $x = y = 0$ at $t = 0$. **Ans.** $x = \dfrac{1}{2} - \dfrac{1}{5}\,e^{-t} - \dfrac{3}{10}\,e^{-\frac{6}{11}t},\ \ y = \dfrac{1}{5}\,e^{-t} - \dfrac{1}{5}\,e^{-\frac{6}{11}t}$

3. $\dfrac{dx}{dt} + 5\,x - 2\,y = t,\ \ \dfrac{dy}{dt} + 2\,x + y = 0$ being given $x = y = 0$ when $t = 0$.

Ans. $x = -\dfrac{1}{27}\,(1+6\,t)\,e^{-3t} + \dfrac{1}{27}\,(1+3\,t),\ \ y = -\dfrac{2}{27}\,(2+3\,t)\,e^{-3t} - \dfrac{2\,t}{9} + \dfrac{4}{27}$

4. $\dfrac{dx}{dt} + y = \sin t,\ \ \dfrac{dy}{dt} + x = \cos t$

given that $x = 2$, and $y = 0$ when $t = 0$. **Ans.** $x = e^t + e^{-t},\ \ y = e^{-t} - e^t + \sin t$

5. $(D-1)\,x-2\,y = t, \quad -2\,x+(D-1)\,y = t \qquad t>0$

where $D = d/dt$ and $x(0) = 2,\ y(0) = 4$

6. The small oscillations of a certain system with two degrees of freedom are given by the equations

$$D^2x+3\,x-2\,y = 0, \quad D^2x+D^2y-3\,x+5\,y = 0$$

If $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$.

Ans. $x = -\frac{11}{4}\sin t+\frac{1}{12}\sin 3\,t, \quad y = \frac{11}{4}\sin t-\frac{1}{4}\sin 3\,t.$

7. $3\frac{dx}{dt}+3\frac{dy}{dt}+5\,x = 25\cos t, \quad 2\frac{dx}{dt}-3\frac{dy}{dt} = 5\sin t$ with $x(0) = 2,\ y(0) = 3.$

Ans. $x = 2\cos t+3\sin t, \quad y = 3\cos t+2\sin 2\,t$

METHODS TO FIND OUT RESIDUES ON PAGE 586 (Art. 7.43)

13.32 INVERSION FORMULA FOR THE LAPLACE TRANSFORM

$f(x)$ = sum of the residues of $e^{sx}F(s)$ at the poles of $F(s)$.

Proof. The Laplace Transform of $f(x)$ is defined by

$$F(s) = \int_0^{\infty} e^{-st}\,.\,f(t)\,dt$$

Multiplying by e^{sx}

$$e^{sx}F(s) = e^{sx}\int_0^{\infty} e^{-st}\,.\,f(t)\,dt$$

Integrating w.r.t. 's' between the limits $a+ir$ and $a-ir$, we have

$$\int_{a-ir}^{a+ir} e^{sx}F(x)\,ds = \int_{a-ir}^{a+ir} e^{sx}\,ds\int_0^{\infty} e^{-st}\,.\,f(t)\,dt$$

Putting $s = a-ip,\ ds = -i\,dp$ $\quad = -i\int_r^{-r} e^{x(a-ip)}\int_0^{\infty} f(t)\,e^{-(a-ip)\,t}\,dt\,dp$

$$= ie^{ax}\int_{-r}^{r} e^{-ipx}\,dp\int_0^{\infty} f(t)\,e^{-at}\,.\,e^{ipt}\,dt. \qquad ...(1)$$

Let us now define $\phi(x)$ as

$$\phi(x) = \begin{cases} e^{-ax}f(x) & \text{when } x\geq 0 \\ 0 & \text{when } x<0 \end{cases}$$

The Fourier complex integral of $\phi(x)$ is

$$\phi(x) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{ipx}\int_{-\infty}^{\infty}\phi(t)\,e^{ipt}\,dt\,dp$$

or

$$e^{-ax}f(x) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{-ipx}\int_0^{\infty}[e^{-at}f(t)]\,e^{ipt}\,dt\,dp \qquad ...(2)$$

In the limiting case when $r\to\infty$, (1) becomes

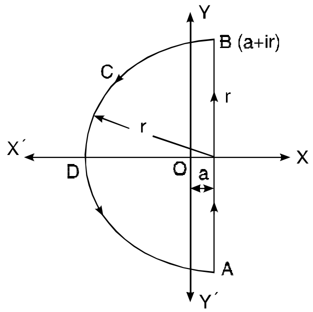
$$\int_{a-i\infty}^{a+i\infty} e^{sx}F(s)\,ds = ie^{ax}\int_{-\infty}^{\infty} e^{-ipx}\,dp\int_{-\infty}^{\infty} f(t)\,e^{-at}\,.\,e^{ipt}\,dt \qquad ...(3)$$

Substituting the value of the integral from (2) in (3), we get

$$\int_{a-i\infty}^{a+i\infty} e^{sx}F(s)\,ds = ie^{ax}[2\,\pi\,e^{-ax}f(x)] = 2\,\pi\,i\,f(x)$$

or

$$f(x) = \frac{1}{2\,\pi\,i}\int_{a-i\infty}^{a+i\infty} e^{sx}F(s)\,ds \qquad ...(4)$$



Equation (4) is called the inversion formula for the Laplace transform.

To obtain $f(x)$, the integration is performed along a line AB parallel to imaginary axis in the complex plane such that all the singularities of $F(s)$ lie to its left. The contour c includes the line AB and the semicircle c' (*i.e.* BDA).

From (4)

$$f(x) = \frac{1}{2\,\pi\,i}\int_{AB} e^{sx}F(x)\,ds$$

$$= \frac{1}{2\,\pi\,i}\int_c e^{sx}F(s)\,ds$$

$$-\frac{1}{2\,\pi\,i}\int_{c'} e^{sx}F(s)\,ds$$

The integral over c' tends to zero as $r\to\infty$. Therefore,

$$f(x) = \lim_{r\to\infty}\frac{1}{2\,\pi\,i}\int_c e^{sx}F(s)\,ds$$

$f(x)$ = sum of the residue of $e^{sx}F(s)$ at the poles of $F(s)$.

Note. Methods for finding the residue: See article 7.43, Chapter 7 on page 586.

Example 61. *Obtain the inverse Laplace transform of* $\frac{s+1}{s^2+2\,s}$.

Solution. Let $\qquad F(s) = \frac{s+1}{s^2+2\,s} \qquad ...(1)$

$$L^{-1}\left[\frac{s+1}{s^2+2\,s}\right] = \text{Sum of the residues of } e^{st}\,.\,\frac{s+1}{s^2+2\,s} \text{ at the poles.} \qquad ...(2)$$

The poles of (1) are determined by equating the denominator to zero, *i.e.*

$$s^2+2\,s = 0 \quad\text{or}\quad s\,(s+2) = 0 \qquad i.e.\ s = 0, -2$$

There are two simple poles at $s = 0$ and $s = -2$.

Residue of $e^{st}\,.\,F(s)$ (at $s = 0$) $= \lim_{s\to 0}\left[(s-0)\frac{e^{st}\,.\,(s+1)}{s^2+2\,s}\right] = \lim_{s\to 0}\left[\frac{e^{st}\,(s+1)}{(s+2)}\right] = \frac{1}{2}$

Residue of $e^{st}\,.\,F(s)$ (at $s = -2$) $= \lim_{s\to -2}\left[\frac{(s+2)\,e^{st}\,(s+1)}{s\,(s+2)}\right]$

$$= \lim_{s\to -2}\left[\frac{e^{st}\,(s+1)}{s}\right] = \frac{e^{-2\,t}\,(-2+1)}{-2} = \frac{e^{-2\,t}}{2}$$

Sum of the residue [at $s = 0$ and $s = -2$] $= \frac{1}{2}+\frac{e^{-2t}}{2}$

Putting the value of residues in (2) we get

$$L^{-1}\left[\frac{s+1}{s^2+2\,s}\right] = \frac{1}{2}+\frac{e^{-2\,t}}{2}$$

Ans.

Example 62. *Find the inverse Laplace transform of* $\dfrac{1}{(s+1)(s^2+1)}$.

Solution. Let
$$F(s) = \frac{1}{(s+1)(s^2+1)} \qquad \text{...(1)}$$
$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \text{sum of residues of } e^{st} F(s) \text{ at the poles.} \qquad \text{...(2)}$$
The poles of (1) are obtained by equating the denominator equal to zero, *i.e.,*
$$(s+1)(s^2+1) = 0 \quad \text{or} \quad s = -1, +i, -i$$
There are three poles of $F(s)$ at $s = -1, s = +i$ and $s = -i$.
$$\text{Residue of } e^{st}.F(s) \text{ (at } s = -1) = \lim_{s\to -1}(s+1)\frac{e^{st}}{(s+1)(s^2+1)} = \lim_{s\to -1}\frac{e^{-t}}{s^2+1} = \frac{e^{-t}}{2}$$
$$\text{Residue of } e^{st}.F(s) \text{ (at } s = i) = \lim_{s\to i}(s-i)\frac{e^{st}}{(s+1)(s^2+1)}$$
$$= \lim_{s\to i}\frac{e^{st}}{(s+1)(s+i)} = \frac{e^{it}}{(i+1)(2i)} = -i\frac{e^{it}}{2}.\frac{1-i}{(1+i)(1-i)} = -\frac{e^{it}}{4}(1+i)$$
$$\text{Residue of } e^{st}.F(s) \text{ (at } s = -i) = \lim_{s\to -i}(s+i)\frac{e^{st}}{(s+1)(s^2+1)}$$
$$= \lim_{s\to -i}\frac{e^{st}}{(s+1)(s-i)} = \frac{e^{-it}}{(-i+1)(-2i)} = \frac{e^{-it}(i-1)}{4}$$
Substituting the values of the residues in (2) we get
$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \frac{e^{-t}}{2} - \frac{e^{it}(1+i)}{4} + \frac{e^{-it}(i-1)}{4}$$
$$= \frac{e^{-t}}{2} + \frac{-e^{it} - ie^{it} + ie^{-it} - e^{-it}}{4} = \frac{e^{-t}}{2} - \frac{e^{it}+e^{-it}}{4} - \frac{i}{2}\,\frac{e^{it}-e^{-it}}{2}$$
$$= \frac{e^{-t}}{2} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$
Ans.

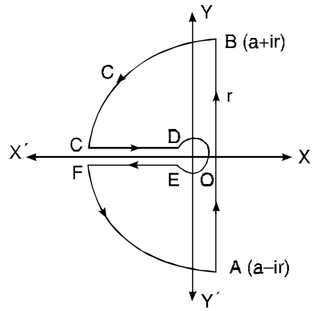
Example 63. *Find the inverse Laplace transform of* $\dfrac{s^2-1}{(s^2+1)^2}$.

Solution. Let
$$F(s) = \frac{s^2-1}{(s^2+1)^2} \qquad \text{...(1)}$$
$$L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = \text{sum of residues of } e^{st}.F(s) \text{ at the poles} \qquad \text{...(2)}$$
The poles of (1) are obtained by equating denominator to zero.
$$(s^2+1)^2 = 0 \quad i.e., \; s = i, -i$$
There are two poles of second order at $s = i$ and $s = -i$.
$$\text{Residue of } e^{st}.F(s) \text{ (at } s = i) = \frac{d}{ds}\left[(s-i)^2\frac{e^{st}(s^2-1)}{(s^2+1)^2}\right]_{s=i} = \frac{d}{ds}\left[\frac{e^{st}(s^2-1)}{(s+i)^2}\right]_{s=i}$$
$$= \left[\frac{(s+i)^2\;[e^{st}.t(s^2-1)+e^{st}2s] - 2(s+i)e^{st}(s^2-1)}{(s+i)^4}\right]_{s=i}$$
$$= \left[\frac{(s+i)[e^{st}.t(s^2-1)+e^{st}.2s]+e^{st}(s^2-1)}{(s+i)^3}\right]_{s=i}$$

$$= \frac{2i[e^{it}.t(-2)+e^{it}2i] - 2e^{it}(-2)}{(2i)^3} = \frac{-4ite^{it}}{-8i} = \frac{te^{it}}{2}$$
$$\text{Residue of } e^{st}.F(s) \text{ (at } s = -i) = \frac{d}{ds}\left[(s+i)^2.\frac{e^{st}(s^2-1)}{(s^2+1)^2}\right]_{s=-i} = \frac{d}{ds}\left[\frac{e^{st}.(s^2-1)}{(s-i)^2}\right]_{s=-i}$$
$$= \left[\frac{(s-i)^2[e^{st}.t(s^2-1)+2se^{st}] - e^{st}(s^2-1)2(s-i)}{(s-i)^4}\right]_{s=-i}$$
$$= \left[\frac{(s-i)[e^{st}.(s^2-1)+2se^{st}] - e^{st}(s^2-1)\;2}{(s-i)^3}\right]_{s=-i}$$
$$= \frac{-2i[\;e^{-it}.t(-2)-2ie^{-it}] - e^{-it}(-2)2}{(-2i)^3} = \frac{4it.e^{-it}}{(-2i)^3} = \frac{t.e^{-it}}{2}$$
Sum of the residues at ($s = i$ and $s = -i$)
$$= \frac{t.e^{it}}{2} + \frac{t.e^{-it}}{2} = t\frac{e^{it}+e^{-it}}{2} = t\cos t. \qquad \text{... (3)}$$
Putting the value of sum of residues from (3) in (2), we get
$$L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = t\cos t$$
Ans.

Example 64. *Obtain the inverse Laplace Transform of* $\dfrac{e^{-b\sqrt{s}}}{s}$.

Solution. Let
$$F(s) = \frac{e^{-b\sqrt{s}}}{s} \qquad \text{...(1)}$$
$$L^{-1}\left(\frac{e^{-b\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}.\frac{e^{-b\sqrt{s}}}{s}\,ds. \qquad \text{...(2)}$$
The simple pole of $F(s)$ is at $s = 0$. Let us have a contour $ABCDEF$ excluding the pole at $x = 0$. The contour encloses no singularity, therefore, by Cauchy theorem



$$\int_{ABCDEFA} e^{st}.F(s) = 0$$
or
$$\int_{AB} e^{st}.F(s)\,ds + \int_{BC} e^{st}.F(s)\,ds + \int_{CD} e^{st}.F(s)\,ds + \int_{DE} e^{st}.F(s)\,ds + \int_{EF} e^{st}.F(s)\,ds + \int_{FA} e^{st}.F(s)\,ds = 0. \qquad \text{...(3)}$$
Let $OC = \rho, OD = \varepsilon$, then along $CD, s = Re^{i\pi}$
$$\int_{CD} e^{sx}.F(s)\,ds = \int_{p}^{\varepsilon} e^{-xR}\frac{e^{-ib\sqrt{r}}}{R}\,dR$$
$$\int_{EF} e^{sx}.F(s)\,ds = \int_{\varepsilon}^{P} e^{-xR}\frac{e^{ib\sqrt{R}}}{R}\,dR \qquad (S = Re^{-i\pi} \text{ along } EF)$$
$$\int_{DE} e^{sx}.F(s)\,ds = \int_{\pi}^{-\pi}\frac{1}{\varepsilon e^{i\theta}}(\varepsilon e^{i\theta}\,id\theta) \qquad \begin{cases} S = \varepsilon e^{i\theta} \text{ along } DE \\ e^{\pi S} = 1 \\ e^{-b\sqrt{s}} = 1 \end{cases}$$

$$= -2\pi i$$

$$\int_{BC} e^{sx} . F(s)\, ds = 0, \quad \int_{FA} e^{sx} . F(s)\, ds = 0$$

On putting the values of the integrals in (3), we have

$$\int_{a-ir}^{a+ir} \frac{e^{xS - b\sqrt{s}}}{s} ds + \int_{\varepsilon}^{P} e^{-xR} \frac{e^{ib\sqrt{R}} - e^{-ib\sqrt{R}}}{R} dR - 2\pi i = 0$$

or $$\int_{a-i\infty}^{a+i\infty} \frac{e^{xS - b\sqrt{s}}}{s} ds = 2\pi i - 2i \int_0^{\infty} e^{-xR} \frac{\sin b\sqrt{R}}{R} dR \qquad \begin{pmatrix} \varepsilon \to 0 \\ p \to \infty \end{pmatrix}$$

or $$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs - b\sqrt{s}}}{s} ds = 1 - \frac{2}{\pi} \int_0^{\infty} e^{-u^2} \frac{\sin\left(\frac{bu}{\sqrt{x}}\right)}{u} du \qquad \left(R = \frac{u^2}{x}\right) \quad ...(4)$$

We know $\displaystyle\int_0^{\infty} e^{-u^2} \cos 2\, bu\, du = \frac{1}{2}\sqrt{\pi}\, e^{-b^2}$

Integrating both sides w.r.t., "b"

$$\int_0^{\infty} e^{-u^2} \left[\frac{\sin 2\, bu}{2\, u}\right] du = \frac{1}{2}\sqrt{\pi} \int e^{-b^2} db$$

Taking limits 0 to $\dfrac{b}{2\sqrt{x}}$, we have

$$\int_0^{\infty} e^{-u^2} \left(\frac{\sin 2\, bu}{2\, u}\right)_0^{\frac{b}{2\sqrt{x}}} du = \frac{\sqrt{\pi}}{2} \int_0^{\frac{b}{2\sqrt{x}}} e^{-b^2} db$$

$$\int_0^{\infty} e^{-u^2} \frac{\sin \frac{bu}{\sqrt{x}}}{u} du = \sqrt{\pi} . \frac{\sqrt{x}}{2} e.r.f. \left(\frac{b}{2\sqrt{x}}\right) \qquad \left[e.r.f.\, x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du\right]$$

$$= \frac{\pi}{2} e.r.f. \left(\frac{b}{2\sqrt{x}}\right)$$

Putting the value of the above integral in (4) we have

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xs} \frac{e^{-b\sqrt{s}}}{s} ds = 1 - \frac{2}{\pi}\frac{\pi}{2} e.r.f. \left(\frac{b}{2\sqrt{x}}\right)$$

$$= 1 - e.r.f. \left(\frac{b}{2\sqrt{x}}\right)$$ **Ans.**

EXERCISE 13.18

Find the inverse of the following by convolution theorem

1. $\dfrac{s^2}{(s^2+a^2)^2}$ **Ans.** $\dfrac{1}{2}\left[t \cos at + \dfrac{1}{2a} \sin at\right]$

2. $\dfrac{1}{s(s^2+a^2)}$ **Ans.** $\dfrac{1 - \cos at}{a^2}$

3. $\dfrac{1}{(s^2+1)^3}$ **Ans.** $\dfrac{1}{8}[(3 - t^2) \sin t - 3t \cos t]$

4. $\dfrac{s}{(s^2+a^2)^2}$ **Ans.** $\dfrac{1}{2a} t \sin at$

Find the Laplace transform of the following

5. $e^{ax} J_0(bx)$ **Ans.** $\dfrac{1}{\sqrt{s^2 + 2as + a^2 + b^2}}$

6. $x J_0(ax)$ **Ans.** $\dfrac{s}{(s^2+a^2)^{3/2}}$

7. $x J_1(x)$ **Ans.** $\dfrac{1}{(s^2+1)^{3/2}}$